S₂-NEAR SUBTRACTION SEMIGROUPS

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ABSTRACT

A near subtraction semigroup X is said to be 1S₂ if for every 0 ∈ X, there exists x ∈ X such that axa = xa. Closely following this, we introduce in this paper the concept of 2S₂-near subtraction semigroups. A near subtraction semigroup X is said to be an 2S₂-near subtraction semigroup if, for every 0 ∈ X, there exists x ∈ X such that axa = xa. Further by generalizing this, we introduce strong 2S₂-near subtraction semigroups. That is, Near subtraction semigroups in which aba = ab for all a, b ∈ X. We also discuss some of their properties.

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1.1 INTRODUCTION

A non empty set X together with a binary operation ‘ − ’ is said to be a subtraction algebra if it satisfies the following axioms:
(i) x − (y − x) = x
(ii) x − (x − y) = y − (y − x)
(iii) (x − y) − z = (x − z) − y for every x, y, z ∈ X.

A non empty set X together with two binary operations ‘ − ’ and ‘ · ’ is said to be a right near subtraction semigroup if it satisfies the following:
(i) (X, − ) is a subtraction algebra.
(ii) (X, ·) is a semigroup.
(iii) (x − y) · z = x · z − y · z for all x, y, z ∈ X.

We shall henceforth write xy for x · y for any two elements x, y of X.

Throughout this paper, X stands for a (right) near subtraction semigroup (X, − , ·) with at least two elements. The subtraction determines an order relation on X: a ≤ b ⇔ a − b = 0 where 0 = a − a is an element that does not depend on the choice of a ∈ X. In X, 0 − x = 0 and 0x = 0 for all x ∈ X.

As in [5] and [6] we define the following: A near subtraction semigroup X is said to have (i) IFP (Insertion of Factors Property) if for a, b in X, ab = 0 ⇒ aba = 0 for all x ∈ X (ii) ( *, IFP) if X has IFP and ab = 0 ⇒ ba = 0 for a, b ∈ X (iii) strong IFP if for all ideals I of X, xy ∈ I ⇒ xny ∈ I for all n in X.

For A, B ⊂ X we define, AB = {ab / a ∈ A, b ∈ B}. We say that a subset Y of X which is closed under ‘ − ’ and XY ⊂ Y is an X-system and if in addition YX ⊂ Y it is called an invariant X-system.

As in [6] and [9], if there exists a map f : X → X such that a = af(a)a for all a in X, then f is called a mate function for X. We say that X is an S(S′) near subtraction semigroup if a ∈ Xa(aX) for all a ∈ X.

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1.2 NOTATIONS

a) An element \( e \in X \) is said to be (i) idempotent if \( e^2 = e \) (ii) nilpotent if \( e^k = 0 \) for some positive integer \( k \) (iii) left identity \( e.a = a \) for every \( a \in X \).

b) \( E \) denotes the set of all idempotents of \( X \).

c) \( L \) denotes the set of all nilpotent elements of \( X \).

d) If \( a^2 = 0 \Rightarrow a = 0 \) for all \( a \in X \), then \( X \) has no non-zero nilpotent elements, as in Problem 14 p-9 of [4].

e) \( X_d = \{ n \in X / n(x - y) = nx - ny, \text{ for all } x, y \in X \} \) - the set of all distributive elements of \( X \).

f) The centre of \( X \) is defined as \( xa = ax \) for all \( x \in X \).

For definitions and notations used but left undefined in this paper we refer to Pilz [5].

2. \( S_2 \)-NEAR SUBTRACTION SEMIGROUPS

Let us now give the definition of \( S_2 \)-near subtraction semigroups.

Definition 2.1: We say that \( X \) is an \( S_2 \)-near subtraction semigroup if for every \( a \in X \) there exists \( *Xa \in X \) such that \( ax = xa \).

Example 2.2: (i) Let \( X = \{0, a, b, 1\} \) in which ‘ − ’ and ‘ ⋅ ’ are defined by

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This is an \( S_2 \)-near subtraction semigroup. [It may be seen that, \( aba = ab, bab = ba, 0a0 = 0a, 1b1 = 1b \)].

(ii) Every Boolean near subtraction semigroup is an \( S_2 \)-near subtraction semigroup.

Proposition 2.3: Every nil near subtraction semigroup is an \( S_2 \)-near subtraction semigroup.

Proof: Let \( X \) be a nil near subtraction semigroup and let \( a \in X^* \). Then there exists a positive integer \( k > 1 \) such that \( a^k = 0 \). We set \( x = a^{k-1} \neq 0 \). Therefore \( ax = 0 \). Now \( ax = (ax)x = 0a = 0 = ax \). That is \( ax = ax \). Clearly \( 0x0 = 0x \) for any \( x \in X^* \). Hence \( X \) is \( S_2 \)-near subtraction semigroup.

Remark 2.4: Converse of Proposition 2.3 is not valid. For example, we consider the near subtraction semigroup \((X, \cdot, \cdot\cdot)\) where \( X = \{0, a, b, c\} \) in which ‘ − ’ and ‘ ⋅ ’ are defined as follows:

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This is an \( S_2 \)-near subtraction semigroup. But it is not a nil near subtraction semigroup. [Since \( c^4 \neq 0 \), for any positive integer \( k \)].

Proposition 2.5: Let \( X \) be an \( S_2 \)-near subtraction semigroup. If \( X \) has no non-zero zero divisors then the following are true.

(i) Every ideal of \( X \) is an \( S_2 \)-near subtraction semigroup.

(ii) Every \( X \)-system of \( X \) is an \( S_2 \)-near subtraction semigroup, in their own right.

Proof: (i) Let \( I \) be an ideal of \( X \) and let \( i \) be a non-zero element of \( I \). Since \( X \) is an \( S_2 \)-near subtraction semigroup, there exists \( y \in X^* \) such that, \( iy = iy \) (1)
If we set \( n = iy \) clearly \( n \in I \). It follows from the hypothesis that \( n \neq 0 \). Now \( ini = i(iy) = i(iyi) = i(iy) \) (by (1)) = \( in \). That is \( ini = in \). Consequently \( I \) is an \( S_2 \)-near subtraction semigroup.

(ii) Let \( M \) be an \( X \)-system of \( X \) and let \( m \) be a non-zero element of \( X \). Since \( X \) is an \( S_2 \)-near subtraction semigroup, there exists \( z \in X \) such that,

\[
mzm = mz = \text{by (2)}.
\]

If we set \( zmd = \) then \( m \in X \). Since \( M \) is an \( X \)-system of \( X \) we get \( Md \in X \). Since \( X \) has no non-zero zero divisors it follows that \( d \neq 0 \). Now \( mzm = (zm)m = (mzm)m = (mz)m = m(zm) = md \). That is \( mzm = md \). Consequently \( M \) is an \( S_2 \)-near subtraction semigroup.

3. STRONG \( S_2 \)-NEAR SUBTRACTION SEMIGROUPS

In this section, we define Strong \( S_2 \)-near subtraction semigroups and obtain its properties.

**Definition 3.1:** We say that \( X \) is a strong \( S_2 \)-near subtraction semigroup if

\[
aba = ab \quad \text{for all } a, b \in X.
\]

**Example 3.2:**

(i) Let \( X = \{0, a, b, 1\} \) in which ‘−’ and ‘⋅’ are defined by

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This is a strong \( S_2 \)-near subtraction semigroup.

(ii) Let \( X = \{0, a, b, c\} \) in which ‘−’ and ‘⋅’ are defined by

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This is a not strong \( S_2 \)-near subtraction semigroup. It is worth noting that it is not zero symmetric.

**Proposition 3.3:** Every strong \( S_2 \)-near subtraction semigroup is an \( S_2 \)-near subtraction semigroup.

**Proof:** Follows from Definitions 3.1 and 2.1.

**Remark 3.4:** Converse of Proposition 3.3 is not valid. For an example, we consider near subtraction semigroup \( X = \{0, a, b, c\} \) where we define ‘−’ and ‘⋅’ as follows:

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This is an \( S_2 \)-near subtraction semigroup. But it is not a strong \( S_2 \)-near subtraction semigroup. (Since \( ac = a \neq ac \)).

**Theorem 3.5:** Let \( X \) be a strong \( S_2 \)-near subtraction semigroup. Then the following are equivalent.

(i) \( X \) is an \( S \)-near subtraction semigroup.
(ii) \( X \) is an \( S' \)-near subtraction semigroup.
(iii) \( X \) is Boolean.
(iv) \( X \) admit mate function.

**Proof:** Since \( X \) is a strong \( S_2 \)-near subtraction semigroup, \( aba = ab \) for all \( a, b \in X \).
(i) $\Rightarrow$ (ii): Let $x \in X$. Since $X$ is an $S$-near subtraction semigroup, there exists $y \in X$ such that $x = xy$. This implies that $xy = (yx)y = yx = x$. That is $x = xy \in xX$. Thus $X$ is an $S'$-near subtraction semigroup.

(ii) $\Rightarrow$ (iii): Let $x \in X$. Since $X$ is an $S'$-near subtraction semigroup, $x \in xX$. Then there exists $y \in X$ such that $x = xy$. This implies $x^2 = (xy)x = xyx = xy = x$. That is $x^2 = x$. Thus $X$ is Boolean.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Obvious.

**Proposition 3.6**: If $X$ is a strong $S_2$-near subtraction semigroup, then $ab$ and $Eba \in E$ for all $X_{ba} \in X$.

**Proof**: Since $X$ is a strong $S_2$-near subtraction semigroup, $xy = xy$ for all $x, y \in X$. Let $a, b \in X^*$. Now $(ab)^2 = abab = a(ba) = a(ba) = aba = ab$. That is $(ab)^2 = ab \Rightarrow ab \in E$. Consequently $ab \in E$ for all $a, b \in X$. In the similar fashion we get $ba \in E$ for all $a, b \in X$.

**Theorem 3.7**: Let $X$ be a strong $S_2$-near subtraction semigroup. Then the following are true.

(i) Every left identity of $X$ is a right identity of $X$.

(ii) $xy$ is a right identity if and only if $x$ and $y$ are right identities for all $X_{xy} \in X$.

(iii) If $0 : xy = 0$ then $xy$ is a right identity for all $x, y \in X$.

**Proof**: Since $X$ is a strong $S_2$-near subtraction semigroup, $xy = xy$ for all $x, y \in X$. Let $e$ be the left identity of $X$ then $en = n$ for all $n \in X$. This implies $(en)e = ne \Rightarrow ene = ne \Rightarrow en = ne \Rightarrow n = ne$. That is $ne = n$ for all $n \in X$. Thus $e$ is the right identity of $X$.

(ii) Let $x, y \in X$. Assume that $xy$ is a right identity. Therefore $nxy = n$ for all $n \in X$. This implies $(nxy)x = nx \Rightarrow n(xy)x = nx \Rightarrow n(xy) = nx \Rightarrow n = nx$. That is $nx = n$. Since $(nx)y = n$, it follows that $ny = n$. Thus $x$ and $y$ are right identities.

Conversely, assume that $x$ and $y$ are right identities. Therefore $zx = z$ and $zy = z$ for all $z \in X$. Now $z(xy) = (zx)y = z$. That is $z(xy) = z$ for all $z \in X$. Thus $xy$ is a right identity.

(iii) Let $z \in X$. Now $(z - zxy)xy = zxy - z(xy)^2 = zxy - zxy$ [since $xy \in E$ by Proposition 3.6] $= 0$. That is $(z - zxy)xy = 0$. Therefore we get, $z - zxy \in (0 : xy)$. Since $(0 : xy) = (0), z - zxy = 0 \Rightarrow z = zxy$. That is $zxy = z$. Thus $xy$ is a right identity of $X$.

**REFERENCES**


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