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S_2-NEAR SUBTRACTION SEMIGROUPS

[1]S. Seyadali Fathima* & [2]R. Balakrishnan

[1]Department of Mathematics, Sadakathullah Appa College, Tirunelveli – 627 002, Tamilnadu, India
[2]Department of Mathematics, V.O. Chidambaram College, Thoothukudi-628 008, Tamilnadu, India

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ABSTRACT

A near subtraction semigroup X is said to be S_1 [10] if for every $a \in X$, there exists $x \in X^* = X - \{0\}$ such that $axa = xa$. Closely following this, we introduce in this paper the concept of S_2-near subtraction semigroups. A near subtraction semigroup X is said to be an S_2-near subtraction semigroup if, for every $a \in X$, there exists $x \in X^*$ such that $axa = ax$. Further by generalizing this, we introduce strong S_2-near subtraction semigroups. That is, Near subtraction semigroups in which $aba = ab$ for all $a, b \in X$. We also discuss some of their properties.

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1.1 INTRODUCTION

A non empty set X together with a binary operation '$-$' is said to be a subtraction algebra if it satisfies the following axioms:
(i) $x-(y-x)=x$
(ii) $x-(x-y)=y-(y-x)$
(iii) $(x-y)-z=(x-z)-y$ for every $x, y, z \in X$.

A non empty set X together with two binary operations '$-$' and '$\cdot$' is said to be a right near subtraction semigroup if it satisfies the following:
(i) $(X,-)$ is a subtraction algebra.
(ii) $(X,\cdot)$ is a semigroup.
(iii) $(x-y)\cdot z = x\cdot z - y\cdot z$ for all $x, y, z \in X$.

We shall henceforth write xy for $x\cdot y$ for any two elements x, y of X.

Throughout this paper, X stands for a (right) near subtraction semigroup $(X,-,\cdot)$ with at least two elements. The subtraction determines an order relation on $X : a \le b \Leftrightarrow a-b=0$ where $0 = a-a$ is an element that does not depend on the choice of $a \in X$. In X, $0-x=0$ and $0x=0$ for all $x \in X$.

As in [5] and [6] we define the following: A near subtraction semigroup X is said to have (i) IFP (Insertion of Factors Property) if for a, b in X, $ab=0 \Rightarrow axb=0$ for all $x \in X$ (ii) $(*, IFP)$ if X has IFP and $ab=0 \Rightarrow ba=0$ for $a, b \in X$ (iii) strong IFP if for all ideals I of X, $xy \in I \Rightarrow xny \in I$ for all n in X.

For $A, B \subset X$ we define, $AB = \{ab \,/\, a \in A, b \in B\}$. we say that a subset Y of X which is closed under '$-$' and $XY \subset Y$ is an X-system and if in addition $YX \subset Y$ it is called an invariant X-system.

As in [6] and [9], if there exists a map $f : X \to X$ such that $a = af(a)a$ for all a in X, then f is called a mate function for X. We say that X is an $S(S')$ near subtraction semigroup if $a \in Xa(aX)$ for all $a \in X$.

Corresponding author: [1]*S. Seyadali Fathima*
[1]Department of Mathematics, Sadakathullah Appa College, Tirunelveli – 627 002, Tamilnadu, India

1.2 NOTATIONS

a) An element $e \in X$ is said to be (i) idempotent if $e^2 = e$ (ii) nilpotent if $e^k = 0$ for some positive integer k (iii) left identity $e.a = a$ for every $a \in X$.

b) E denotes the set of all idempotents of X.

c) L denotes the set of all nilpotent elements of X.

d) If $a^2 = 0 \Rightarrow a = 0$ for all $a \in X$, then X has no non-zero nilpotent elements, as in Problem 14 p-9 of [4].

e) $X_d = \{n \in X \ / \ n(x-y) = nx - ny,$ for all $x, y \in X\}$ - the set of all distributive elements of X.

f) The centre of X is defined as $C(X) = \{a \in X \ / \ ax = xa$ for all $x \in X\}$.

For definitions and notations used but left undefined in this paper we refer to Pilz [5].

2. S_2-NEAR SUBTRACTION SEMIGROUPS

Let us now give the definition of S_2-near subtraction semigroups.

Definition 2.1: We say that X is an S_2-near subtraction semigroup if for every $a \in X$, there exists $x \in X^*$ such that $axa = ax$.

Example 2.2: **(i)** Let $X = \{0, a, b, 1\}$ in which '$-$' and '$\cdot$' are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

.	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

This is an S_2-near subtraction semigroup. [It may be seen that, $aba = ab, bab = ba, 0a0 = 0a, 1b1 = 1b$].

(ii) Every Boolean near subtraction semigroup is an S_2-near subtraction semigroup.

Proposition 2.3: Every nil near subtraction semigroup is an S_2-near subtraction semigroup.

Proof: Let X be a nil near subtraction semigroup and let $a \in X^*$. Then there exists a positive integer $k > 1$ such that $a^k = 0$. We set $x = a^{k-1} \neq 0$. Therefore $ax = 0$. Now $axa = (ax)x = 0a = 0 = ax$. That is $axa = ax$. Clearly $0x0 = 0x$ for any $x \in X^*$. Hence X is S_2-near subtraction semigroup.

Remark 2.4: Converse of Proposition 2.3 is not valid. For example, we consider the near subtraction semigroup $(X, -, \cdot)$ where $X = \{0, a, b, c\}$ in which '$-$' and '$\cdot$' are defined as follows:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	0	c

This is an S_2-near subtraction semigroup. But it is not a nil near subtraction semigroup. [Since $c^k \neq 0,$ for any positive integer k].

Proposition 2.5: Let X be an S_2-near subtraction semigroup. If X has no non-zero zero divisors then the following are true.

(i) Every ideal of X is an S_2-near subtraction semigroup.

(ii) Every X-system of X is an S_2-near subtraction semigroup, in their own right.

Proof: **(i)** Let I be an ideal of X and let i be a non-zero element of I. Since X is an S_2-near subtraction semigroup, there exists $y \in X^*$ such that,

$$iyi = iy \tag{1}$$

If we set $n = iy$ clearly $n \in I$. It follows from the hypothesis that $n \neq 0$. Now $ini = i(iy)i = i(iyi) = i(iy)$ (by (1)) $= in$. That is $ini = in$. Consequently I is an S_2-near subtraction semigroup.

(ii) Let M be an X-system of X and let m be a non-zero element of X. since X is an S_2-near subtraction semigroup, there exists $z \in X^*$ such that,

$$mzm = mz \qquad (2)$$

If we set $d = zm$ then $d \in XM$. Since M is an X-system of X, we get $d \in M$. Since X has no non-zero zero divisors it follows that $d \neq 0$. Now $mdm = m(zm)m = (mzm)m = (mz)m$ (by (2)) $= m(zm) = md$. That is $mdm = md$. Consequently M is an S_2-near subtraction semigroup.

3. STRONG S_2-NEAR SUBTRACTION SEMIGROUPS

In this section, we define Strong S_2-near subtraction semigroups and obtain its properties.

Definition 3.1: We say that X is a strong S_2-near subtraction semigroup if $aba = ab$ for all $a, b \in X$.

Example 3.2: **(i)** Let $X = \{0, a, b, 1\}$ in which '$-$' and '$\cdot$' are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

.	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

This is a strong S_2-near subtraction semigroup.

(ii) Let $X = \{0, a, b, c\}$ in which '$-$' and '$\cdot$' are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

.	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

This is a not strong S_2-near subtraction semigroup. It is worth noting that it is not zero symmetric.

Proposition 3.3: Every strong S_2-near subtraction semigroup is an S_2-near subtraction semigroup.

Proof: Follows from Definitions 3.1 and 2.1.

Remark 3.4: Converse of Proposition 3.3 is not valid. For an example, we consider near subtraction semigroup $X = \{0, a, b, c\}$ where we define '$-$' and '$\cdot$' as follows:

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
c	0	0	0	c

This is an S_2-near subtraction semigroup. But it is not a strong S_2-near subtraction semigroup. (Since $aca \neq ac$).

Theorem 3.5: Let X be a strong S_2-near subtraction semigroup. Then the following are equivalent.

(i) X is an S-near subtraction semigroup.
(ii) X is an S'-near subtraction semigroup.
(iii) X is Boolean.
(iv) X admit mate function.

Proof: Since X is a strong S_2-near subtraction semigroup, $aba = ab$ for all $a, b \in X$.

(i) ⇒ (ii): Let $x \in X$. Since X is an S-near subtraction semigroup, there exists $y \in X$ such that $x = yx$. This implies that $xy = (yx)y = yx = x$. That is $x = xy \in xX$. Thus X is an S'-near subtraction semigroup.

(ii) ⇒ (iii): Let $x \in X$. Since X is an S'-near subtraction semigroup, $x \in xX$. Then there exists $y \in X$ such that $x = xy$. This implies $x^2 = (xy)x = xyx = xy = x$. That is $x^2 = x$. Thus X is Boolean.

(iii) ⇒ (iv): Obvious.

(iv) ⇒ (i): Obvious.

Proposition 3.6: If X is a strong S_2-near subtraction semigroup, then ab and $ba \in E$ for all $a, b \in X$.

Proof: Since X is a strong S_2-near subtraction semigroup, $xyx = xy$ for all $x, y \in X$. Let $a, b \in X^*$. Now $(ab)^2 = abab = a(bab) = a(ba) = aba = ab$. That is $(ab)^2 = ab \Rightarrow ab \in E$. Consequently $ab \in E$ for all $a, b \in X$. In the similar fashion we get $ba \in E$ for all $a, b \in X$.

Theorem 3.7: Let X be a strong S_2-near subtraction semigroup. Then the following are true.
(i) Every left identity of X is a right identity of X.
(ii) xy is a right identity if and only if x and y are right identities for all $xy \in X$.
(iii) If $(0 : xy) = \{0\}$ then xy is a right identity for all $x, y \in X$.

Proof: Since X is a strong S_2-near subtraction semigroup, $aba = ab$ for all $a, b \in X$.

(i) If e is the left identity of X then $en = n$ for all $n \in X$. This implies $(en)e = ne \Rightarrow ene = ne \Rightarrow en = ne \Rightarrow n = ne$. That is $ne = n$ for all $n \in X$. Thus 'e' is the right identity of X.

(ii) Let $x, y \in X$. Assume that xy is a right identity. Therefore $nxy = n$ for all $n \in X$. This implies $(nxy)x = nx \Rightarrow n(xyx) = nx \Rightarrow n(xy) = nx \Rightarrow n = nx$. That is $nx = n$. Since $(nx)y = n$, it follows that $ny = n$. Thus x and y are right identities.

Conversely, assume that x and y are right identities. Therefore $zx = z$ and $zy = z$ for all $z \in X$. Now $z(xy) = (zx)y = z$. That is $z(xy) = z$ for all $z \in X$. Thus xy is a right identity.

(iii) Let $z \in X$. Now $(z - zxy)xy = zxy - z(xy)^2 = zxy - zxy$ [since $xy \in E$ by Proposition 3.6] $= 0$. That is $(z - zxy)xy = 0$. Therefore we get, $z - zxy \in (0 : xy)$. Since $(0 : xy) = \{0\}$, $z - zxy = 0 \Rightarrow z = zxy$. That is $zxy = z$. Thus xy is a right identity of X.

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