Research Journal of Pure Algebra -2(12), 2012, Page: 365-369
Available online through www.rjpa.info ISSN 2248–9037

SIX MAPS WITH A COMMON FIXED POINT IN COMPLEX VALUED METRIC SPACES

[1]RAHUL TIWARI* & [2]D.P. SHUKLA

[1]Department of Mathematical Sciences, A.P.S. University Rewa, 486003, India
[2]Department of Mathematics, Govt. P.G. Science College Rewa, 486001, India

(Received on: 19-11-12; Revised & Accepted on: 07-12-12)

ABSTRACT

***R**ecently, Sandeep Bhatt et.al [12] proved common fixed theorem for four self maps satisfying some contraction principles on a complex valued metric spaces. In this manuscript we obtain a common fixed point theorem for six maps in complex valued metric spaces having commuting and weakly compatible. Our theorem generalizes and extends the result of S. Bhatt et.al [12].*

Mathematics Subject Classification: *47H10, 54H25.*

Keywords: *weakly compatible maps, fixed points, common fixed points, complex valued metric spaces.*

1. INTRODUCTION

A large variety of the problems of analysis and applied mathematics reduce to finding solutions of non linear functional equations which can be formulated in terms of finding the fixed points of a non-linear mapping. Fixed point theorems are also used to study the problems of optimal control related to systems [4].

The study of metric spaces expressed the most important role to many fields both in pure and applied science such as biology, medicine, physics and computer science (see [14, 3]). Many authors generalized and extended the notion of a metric spaces such as vector-valued metric spaces of Perov [2], a G-metric spaces of Mustafa and Sims [16], a cone metric spaces of Huang and Zhang [10], a modular metric spaces of Chistyakov [15], and etc.

A. Azam, B. Fisher and M. Khan [1] first introduced the complex valued metric spaces which is more general than well-know metric spaces and also gave common fixed point theorems for maps satisfying generalized contraction condition.

2. PRELIMINARIES

Let $\mathbb{C}$ be the set of all complex numbers. For $z_1, z_2 \in \mathbb{C}$, define partial order $\precsim$ on $\mathbb{C}$ by $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

That is $z_1 \precsim z_2$ if one of the following conditions holds

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied and we will write $z_1 \prec z_2$.

Definition 2.1[1] Let X be a non-empty set and $d: X \times X \to \mathbb{C}$ be a map, then d is said to be complex valued metric if

(i) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x=y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Pair (X, d) is called a complex valued metric space.

Example 2.2: Define a map $d: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(z_1, z_2) = e^{ip}|z_1 - z_2|$ where $p \in \mathbb{R}$. Then $(\mathbb{C}, d)$ is a complex valued metric space.

***Corresponding author:** *RAHUL TIWARI**
[1]Department of Mathematical Sciences, A.P.S. University Rewa, 486003, India.
E-mail: tiwari.rahul.rewa@gmail.com

Definition 2.3[1] Let (X, d) be a complex valued metric space then

(i) Any point x in X is said to be an interior point of $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that
$$B(x, r) = \{y \in X \mid d(x, y) \prec r\} \subseteq A.$$
(ii) Any point x in X is said to be a limit point of A if for every $0 \prec r \in \mathbb{C}$, we have $B(x, r) \cap (A - X) \neq \phi$.

(iii) Any subset A of X is said to be an open if each element of A is an interior point of A.

(iv) Any subset A of X is said to be a closed if each limit point of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family given by $F = \{B(x, r) \mid x \in X \text{ and } 0 \prec r\}$.

Definition 2.4 [1]Let $\{x_n\}$ be a sequence in complex valued metric space (X, d) and $x \in X$
Then

(i) It is said to be a convergent sequence, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, if for every $c \in \mathbb{C}$, with $0 \prec c$ there is a natural number N such that $d(x_n, x) \prec c$, for all n>N. We denote it by $\mathrm{Lim}_{n\to\infty} x_n = x$

(ii) It is said to be a Cauchy sequence, if for every $c \in \mathbb{C}$, with $0 \prec c$ there is a natural number N such that $d(x_n, x_{n+m}) \prec c$, for all n>N and $m \in \mathbb{N}$.

(iii) (X, d) is said to be complete complex valued metric space if every Cauchy sequence in X is convergent.

Lemma 2.5 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d), converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.6 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d) is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

Definition 2.7 Let S and T be self maps of a non-empty set X. Then

(i) Any point $x \in X$ is said to be fixed point of T if Tx = x.

(ii) Any point $x \in X$ is said to be a coincidence point of S and T if Sx = Tx and we shall called w = Sx= Tx that a point of coincidence of S and T.

(iii) Any point $x \in X$ is said to be a common fixed point of S and T if Sx =Tx =x

Definition 2.8 [5]: Two self maps S, T of a non-empty set X are commuting if
$$TSx = STx, \text{ for all } x \in X.$$

Definition 2.9 [13]: Let S, T be self maps of metric space (X, d), then S, T are said to be weakly commuting if
$$d(STx, TSx) \leq d(Sx, Tx), \text{ for all } x \in X.$$

Definition 2.10 [6] Let S, T be self maps of metric space (X, d), then S, T are said to be compatible if
$$\mathrm{Lim}_{n\to\infty} d(STx, TSx_n) = 0$$
whenever $\{x_n\}$ is a sequence in X such that $\mathrm{Lim}_{n\to\infty} Sx_n = \mathrm{Lim}_{n\to\infty} Tx_n = z$, for some $z \in X$.

Remark 2.11 In general, commuting maps are weakly commuting and weakly commuting maps are compatible, but the converses are not necessarily true and some examples can be found in **[5-7, 9]**

Definition 2.12 [8] Two self maps S, T of a non-empty set X are said to be weakly compatible if
STx = TSx whenever Sx = Tx.

Lemma 2.13 [11] Let $T: X \to X$ be a map, then there exists a subset E of X such that T(E) = T(X) and $T: E \to X$ is one to one.

3. MAIN RESULT

Theorem 3.1: Let (X, d) be a complex valued metric space and F, G, I, J, K, L be self maps of X satisfying the following conditions

$$KL(X) \subseteq F(X) \text{ and } IJ(X) \subseteq G(X) \tag{3.1}$$

$$d(IJx, KLy) \leq ad(Fx, Gy) + b(d(Fx, IJx) + d(Gy, KLy)) + c(d(Fx, KLy) + d(Gy, IJx)) \tag{3.2}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a+2b+2c < 1$.
Assume that pairs (KL, G) and (IJ, F) are weakly compatible. Pairs (K, L), (K, G), (L, G), (I, J), (I, F) and (J, F) are commuting pairs of maps. Then K, L, I, J, G and F have a unique common fixed point in X.

Proof: Pick $x_o \in X$. By (3.1), we can define inductively a sequence $\{y_n\}$ in X such that
$$y_{2n} = IJx_{2n} = Gx_{2n+1} \text{ and } y_{2n+1} = KLx_{2n+1} = Fx_{2n+2} \text{ for all } n = 1, 2, 3,\ldots \tag{3.3}$$

By (3.2), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(IJx_{2n}, KLx_{2n+1}) \\ &\le a\, d(Fx_{2n}, Gx_{2n+1}) + b(d(Fx_{2n}, IJx_{2n})+d(Gx_{2n+1}, KLx_{2n+1}))+ c(d(Fx_{2n}, KLx_{2n+1})+d(Gx_{2n+1}, IJx_{2n})) \\ &= a\, d(y_{2n-1}, y_{2n}) + b(d(y_{2n-1}, y_{2n})+d(y_{2n}, y_{2n+1}))+ c(d(y_{2n-1}, y_{2n+1})+d(y_{2n}, y_{2n})) \\ &\le (a+b+c)\, d(y_{2n-1}, y_{2n}) + (b+c)\, d(y_{2n}, y_{2n+1}) \end{aligned}$$

which implies that

$$d(y_{2n}, y_{2n+1}) \le \frac{a+b+c}{1-b-c}\, d(y_{2n-1}, y_{2n}) = k\, d(y_{2n-1}, y_{2n})$$

where $k = \dfrac{a+b+c}{1-b-c} < 1$. Similarly we obtain $d(y_{2n+1}, y_{2n+2}) \le k\, d(y_{2n}, y_{2n+1})$

Therefore,

$$d(y_{n+1}, y_{n+2}) \le k\, d(y_n, y_{n-1}) \le \ldots k^{n+1} d(y_o, y_1) \quad \text{for } n = 1, 2, 3, \ldots$$

Now, for all m > n,

$$\begin{aligned} (y_n, y_m) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots d(y_{m-1}, y_m) \\ &\le (k^n + k^{n+1} + \ldots + k^{m-1})\, d(y_1, y_o) \\ &\le \frac{k^n}{k-1} d\left(y_1, y_0\right) \end{aligned}$$

$$\Rightarrow d|(y_n, y_m)| \le \frac{k^n}{k-1}\left|d\left(y_1, y_0\right)\right| \; k^n \quad |d(y_1, y_o)|$$

which implies that $d|(y_n, y_m)| \to 0$ as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence

Since X is complete, there exists a point z in X such that

$$\operatorname{Lim}_{n\to\infty} IJx_{2n} = \operatorname{Lim}_{n\to\infty} Gx_{2n+1} = \operatorname{Lim}_{n\to\infty} KLx_{2n+1} = \operatorname{Lim}_{n\to\infty} Fx_{2n+2} = z$$

Since $KL(X) \subseteq F(X)$, there exists a point $u \in X$ such that $z = Fu$.

Then by (3.2), we have

$$\begin{aligned} d(IJu, z) &\le d(IJu, KLx_{2n-1}) + d(KLx_{2n-1}, z) \\ &\le a\, d(Fu, Gx_{2n-1}) + b(d(Fu, IJu)+d(Gx_{2n-1}, KLx_{2n-1})) + c(d(Fu, KLx_{2n-1})+d(Gx_{2n-1}, IJu)) + d(KLx_{2n-1}, z) \end{aligned}$$

Taking the limit as $n \to \infty$, weobtain

$$\begin{aligned} d(IJu, z) &\le a\, d(z, z) + b(d(z, IJu) + d(z, z)) + c(d(z, z) + d(z, IJu)) + d(z, z) \\ &= (b+c)\, d(IJu, z), \text{ a contradiction} \end{aligned}$$

Since $a+2b+2c < 1$. Therefore $IJu = Fu = z$. Since $IJ(X) \subseteq G(X)$, there exists a point v in X such that $z = Gv$.

Then by (3.2), we have

$$\begin{aligned} d(z, KLv) &= d(IJu, KLv) \\ &\le a\, d(Fu, Gv) + b(d(Fu, IJu) + d(Gv, KLv)) + c(d(Fu, KLv) + d(Gv, IJu)) \\ &= a\, d(z, z) + b(d(z, z) + d(z, KLv)) + c(d(z, KLv) + d(z, z)) \\ &= (b+c)\, d(z, KLv), \text{ which is a contradiction.} \end{aligned}$$

Therefore $KLv = Gv = z$ and so $IJu = Fu = KLv = Gv = z$.

Since F and IJ are weakly compatible maps, $IJFu = FIJu$ and so $IJz = Fz$. Now we claim that z is a fixed point of IJ if $IJz \ne z$, from (3.2), we have

$$\begin{aligned} d(IJz, z) &= d(IJz, KLv) \\ &\le a\, d(Fz, Gv) + b(d(Fz, IJz)+d(Gv, KLv)) + c(d(Fz, KLv)+d(Gv, IJz)) \\ &= a\, d(IJz, z) + b(d(IJz, IJz)+d(z, z) + c(d(IJz, z)+d(z, IJz)) \\ &= (a+2c)\, d(IJz, z), \text{ a contradiction.} \end{aligned}$$

Therefore IJz = z. Hence IJz = Fz = z.

Similarly, G and KL are weakly compatible maps, we have KLz = Gz

Now we claim that z is a fixed point of KL. If KLz $\neq$ z, then by (3.2), we have

$$\begin{aligned} d(z, KLz) &= d(IJz, KLz) \\ &\le a\, d(Fz, Gz) + b(d(Fz, IJz) + d(Gz, KLz) + c(d(Fz, KLz) + d(Gz, IJz)) \\ &= a\, d(z, KLz) + b(d(z, z) + d(KLz, KLz)) + c(d(z, KLz)+d(KLz, z)) \\ &= (a+2c)\, d(z, KLz), \text{ a contradiction.} \end{aligned}$$

Therefore KLz = z. Hence KLz = Gz = z. We have therefore proved that IJz = KLz = Fz = Gz = z. So z is common fixed point of F, G, IJ and KL.

By commuting conditions of pairs we have

Kz = K(KLz) = K(LKz) = KL(Kz).
Kz = K(Fz) = F(Kz) and Lz = L(KLz) = (LK)(Lz) = (KL)(Lz),
Lz = L(Fz) = F(Lz), which shows that Kz and Lz are common fixed points of (KL, F)

Then Kz = z = Lz = Fz = KLz

Similarly Iz = z =Jz = Gz = IJz

Therefore z is a common fixed point of K, L, I, J, F and G.

For uniqueness of z, let w be another common fixed point of K, L, I, J, F and G.

Then by (3.2), we have

$$\begin{aligned} d(z, w) &= d(IJz, KLw) \\ &\le a\, d(Fz, Gw) + b(d(Fz, IJz) + d(Gw, KLw) + c(d(Fz, KLw)+d(Gw, IJz)) \\ &= a\, d(z, w) + b(d(z, z) + d(w, w) + c(d(z, w) + d(w, z)) \\ &= (a+2c)\, d(z, w), \text{ a contradiction.} \end{aligned}$$

So z = w.

REFERENCES

1. A. Azam, B. Fisher and M Khan: Common Fixed Point Theorems in Complex Valued Metric Sapces. Numerical Functional Analysis and Optimization. 32(3): 243-253(2011). doi:10.1080/01630563.2011.533046
2. Al Perov: On the Cauchy problem for a system of ordinary differential equations. Pvi-blizhen Met Reshen Diff Uvavn. 2, 115-134(1964)
3. C. Semple, M. Steel: Phylogenetics, Oxford Lecture Ser. in Math Appl, vol. 24, Oxford Univ. Press, Oxford(2003)
4. D. Wardowski: Endpoints and fixed points of set-valued contractions in cone metric spaces. Nonlinear Analysis, vol. 71, pp. 512-516, 2009.
5. G. Jungck: Commuting maps and fixed points. Am Math Monthly. 83, 261-263(1976). doi:10.2307/2318216
6. G. Jungck: Compatible mappings and common fixed points. Int J Math Math Sci. 9, 771-779(1986). doi:10.1155/S0161171286000935
7. G. Jungck: Common fixed points of commuting and compatible maps on compacta. Proc Am Math Soc. 103, 977-983(1988). doi:10.1090/S0002-9939-1988-0947693-2
8. G. Jungck: Common fixed points of non-continuous non-self mappings on a non-numeric spaces. Far East J Math Sci. 4(2):199-212(1996)
9. G. Jungck: Compatible mappings and common fixed points (2). Int J Math Math Sci. 11, 285-288(1988). doi: 10.1155/S0161171288000341
10. L.G Huang, X. Zhang: Cone metric spaces and fixed point theorems of contractive mappings. J Math Anal Appl. 332, 1468-1476(2007). doi: 10.1016/j.jmaa.2005.03.087
11. R.H Haghi, Sh. Rezapour and N. Shahzadb: Some fixed point generalizations are not real generalization. Nonlinear Anal. 74, 1799-1803(2011). doi:10.1016/j. na.2010.10.052
12. S. Bhatt, S. Chaukiyal and R.C. Dimri: A common fixed point theorem for weakly compatible maps in complex valued metric spaces. Int. J. of Mathematical Sciences and Applications, vol. 1, No. 3(2011).
13. S. Sessa: On a weak commutativity condition of mappings in fixed point consideration. Publ Inst Math, 32 (46): 149- 153(1982)

14. W.A Kirk: Some recent results in metric fixed point theory. J Fixed Point Theory Appl. 2, 195-207 (2007). Doi: 10.1007/s11784-007-0031-8
15. W. Chistyakov, Modular metric spaces, I: basic concepts. Nonlinear Anal. 72, 1-14(2010) doi: 10. 1016/j. na 2009. 04. 057
16. Z. Mustafa, B. Sims: A new approach to generalized metric spaces. J Nonlinear Convex Anal. 7(2):289-297(2006).

Source of support: Nil, Conflict of interest: None Declared