COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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ABSTRACT

Let $R$ be a ring and $M$ be an $R$-module. $M$ is called cofinitely weak Rad-supplemented module if every cofinite submodule of $M$ has a weak Rad-supplement in $M$. If every cofinite submodule of $M$ has ample weak Rad-supplements in $M$, then $M$ is called amply cofinitely weak Rad-supplemented module. In this paper we study some properties of such type of modules.

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Key words: Cofinitely supplemented module, Cofinitely Rad-supplemented module, cofinitely weak Rad-supplemented module, amply cofinitely weak Rad-supplemented module

1. INTRODUCTION

Throughout this paper all rings are associative rings with identity and all modules are unital left $R$-modules. Let $R$ be a ring and $M$ be an $R$-module. The notation $N \subseteq M$ means that $N$ is a submodule of $M$. A submodule $K$ of an $R$-module $M$ is called small in $M$ (denoted by $K \ll M$) if $K+L=M$ for any submodule $L$ of $M$ implies $L=M$, see [1]. $\text{Rad}(M)$ indicates the Jacobson radical of $M$. A module $M$ is called semi-hollow if every finitely generated proper submodule is small in $M$, or $\text{Rad}(M)=M$, see [2]. Let $M$ be an $R$-module and $A$ and $B$ be any submodules of $M$. $B$ is called a supplement of $A$ in $M$ if $B$ is minimal with respect to $M=A+B$. $B$ is a supplement of $A$ in $M$ iff $M=A+B$ and $A \cap B \ll B$, (see [2] 20.1). $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. Artinian and semisimple modules are supplemented modules. For $A \subseteq M$, a submodule $B$ of $M$ is called a weak supplement of $A$ in $M$ if $A+B=M$ and $A \cap B \ll M$ (see [12], 1.3). An $R$-module $M$ is called weakly supplemented if every submodule of $M$ has a weak supplement in $M$. Clearly supplemented modules are weakly supplemented. Artinian, semisimple and hollow modules are weakly supplemented modules.

A submodule $K$ of a module $M$ is said to be cofinite if the factor module $M/K$ is finitely generated. If every cofinite submodule of $M$ has a supplement in $M$ then $M$ is called a cofinitely supplemented module, see [3]. An $R$-module $M$ is called a cofinitely weak supplemented module (or briefly a cws-module) if every cofinite submodule has a weak supplement. Clearly cofinitely supplemented module and weakly supplemented module are cofinitely weak supplemented and a finitely generated proper submodule is small in $M$. Cofinitely supplemented module and weakly supplemented module are cofinitely weak supplemented modules. Cofinitely supplemented modules are weakly supplemented. Artinian, semisimple and hollow modules are weakly supplemented modules.

Let $M$ be an $R$-module and let $U$ be a submodule of $M$. A submodule $V$ of $M$ is called a Rad-supplement of $U$ in $M$ (according to [5], generalized supplement) if $U+V=M$ and $U \cap V \subseteq \text{Rad}(V)$. An $R$-module $M$ is called Rad-supplemented (according to [5], generalized supplemented or a GS-module) if every submodule of $M$ has aRad-supplement in $M$. A submodule $V$ of $M$ is called a weak Rad-supplement of $U$ in $M$ if $U+V=M$ and $U \cap V \subseteq \text{Rad}(M)$. An $R$-module $M$ is called weakly Rad-supplemented (according to [5], generalized weakly supplemented or a WGS-module) if every submodule of $M$ has a weak Rad-supplement in $M$. The $Z$-module $Q$ is Rad-supplemented as well as weak Rad-supplemented modules but the $Z$-module $Q$ is not supplemented. Let $M$ be an $R$-module. If every cofinite submodule of $M$ has a Rad-supplement in $M$ then $M$ is called a cofinitely Rad-supplemented module.

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Let $M$ be an $R$-module and $N \in \sigma [M]$, subcategory of left $R$-modules subgenerated by $M$. A projective module $P$ in $\sigma [M]$ together with a small epimorphism $\pi : P \to N$ is called a projective cover of $N$ in $\sigma [M]$. A module $N$ in $\sigma [M]$ is called semiperfect in $\sigma [M]$ if every factor module of $N$ has a projective cover in $\sigma [M]$. A projective module in $\sigma [M]$ is semiperfect in $\sigma [M]$ if and only if it is (amply) supplemented (see [1], 42.3).

2. COFINITELY WEAK RAD-SUPPLEMENTED MODULE

In this section, we discuss the concept of cofinitely weak Rad-supplemented modules and give some properties of such type of modules.

Definition 2.1. Let $M$ be an $R$-module. $M$ is called a cofinitely weak Rad-supplemented module if every cofinite submodule of $M$ has a weak Rad-supplement in $M$.

Lemma 2.2. Let $M$ be an $R$-module and $V$ be a weak Rad-supplement of $U$ in $M$. Then $(V + L)/L$ is a weak Rad-supplement of $U/L$ in $M/L$ for every submodule $L$ of $U$.

Proof: See [6, Lemma II. 1]

Theorem 2.3. Let $M$ be an $R$-module and $N$ be a nonzero semi-hollow submodule of $M$. Then $M$ is cofinitely weak Rad-supplemented iff $M/N$ is cofinitely weak Rad-supplemented.

Proof: Let $M$ be a cofinitely weak Rad-supplemented module. Let $U$ be a submodule of $M$ and $N$ be a nonzero semi-hollow submodule of $M$. Consider $U/N$ is a cofinite submodule of $M/N$, then $U$ is cofinite. Since $M$ is cofinitely weak Rad-supplemented module, then there is a submodule $V$ of $M$ such that $U + V = M$ with $U \cap V \subseteq \text{Rad} M$. By lemma 2.2, $(V + N)/N$ is a weak Rad-supplement of $U/N$ in $M/N$. Hence $M/N$ is cofinitely weak Rad-supplemented.

Conversely, Let $U$ be a cofinite submodule of $M$. Then $(U + N)/N$ is a cofinite submodule of $M/N$. Since $M/N$ is cofinitely weak Rad-supplemented, $(U + N)/N$ has a weak Rad-supplement in $M/N$. Suppose $V/N$ is weak Rad-supplement of $(U + N)/N$ in $M/N$. Then $V/N + (U + N)/N = M/N \Rightarrow (U + V)/N = M/N \Rightarrow U + V = M$ and $V/N \cap (U + V)/N \subseteq \text{Rad} (M/N) \Rightarrow (U \cap V)/N \subseteq \text{Rad} M \Rightarrow (U \cap V)/N \subseteq \text{Rad} M/N$ (since $N$ is semi-hollow module, so $\text{Rad} N = N$) $\Rightarrow U \cap V \subseteq \text{Rad} M$. Hence $M$ is a cofinitely weak Rad-supplemented.

Proposition 2.4. Suppose that $M$ be a cofinitely weak Rad-supplemented module and $N$ be a submodule with $\text{Rad} M \subseteq N$. If $\text{Rad} (M/N) = \{N\}$, then every cofinite submodule of $M/N$ is a direct summand of $M/N$.

Proof: Let $M$ be a cofinitely weak Rad-supplemented module and $M/N$ be any factor module of $M$. For $N \subseteq K$, let $K/N$ be a cofinite submodule of $M/N$, then $M/N$ is finitely generated. Now $M/K \cong M/N$ $K/N$ therefore, $M/K$ is finitely generated. Hence $K$ is a cofinite submodule of $M$. Since $M$ is cofinitely weak Rad-supplemented module, therefore, there is a submodule $V$ of $M$ such that $K + V = M$ and $K \cap V \subseteq \text{Rad} M$. According to the lemma 2.2, $(V + N)/N$ is a weak Rad-supplement of $K/N$ in $M/N$. Hence $K/N + (V + N)/N = M/N \Rightarrow (K + V)/N = M/N \Rightarrow K + V = M$ and $(V + N)/N \cap K/N \subseteq \text{Rad} (M/N) = \{N\}$. Since $\text{Rad} M \subseteq N$, we know $\{N\} \subseteq (V + N)/N \cap K/N$, therefore we have $(V + N)/N \cap K/N = \{N\}$. Hence $K/N$ is a direct summand of $M/N$.

Corollary 2.5. Let $M$ be a cofinitely weak Rad-supplemented module. Then every cofinite submodule of $M/\text{Rad} M$ is a direct summand of $M/\text{Rad} M$.

Lemma 2.6. If $f : M \to N$ is a homomorphism and a submodule $L$ of $M$ containing $\ker f$ is a weak Rad-supplement in $M$, then $f(L)$ is a weak Rad-supplement in $f(M)$.
**Proof:** Let $M$, $N$ be $R$-modules and $f : M \to N$ be a homomorphism. If $L$ is a weak Rad-supplement of $K$ in $M$, then we have $M = L + K \Rightarrow f(M) = f(L) + f(K)$ and since $L \cap K \subseteq \text{Rad } M$ we have $f(L \cap K) \subseteq f(\text{Rad } M) \subseteq \text{Rad } f(M)$. As $\ker f \subseteq L$, $f(L \cap f(K)) = f(L \cap K)$ i.e. $f(L \cap f(K)) \subseteq \text{Rad } f(M)$. So $f(L)$ is a weak Rad-supplement of $f(K)$ in $f(M)$.

**Proposition 2.7.** Every homomorphic image of cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.

**Proof:** Suppose that $f : M \to N$ be a homomorphism and $M$ be a cofinitely weak Rad-supplemented module. Let $K$ be a cofinite submodule of $f(M)$, then $M / f^{-1}(K) \cong (M / \ker f) / (f^{-1}(K)) / \ker f \cong f(M) / K$. Therefore, $M / f^{-1}(K)$ is finitely generated. Since $M$ is a cofinitely weak Rad-supplemented module, $f^{-1}(K)$ is a weak Rad-supplemented module in $M$ and according to the lemma 2.6, $K = f(f^{-1}(K))$ is a weak Rad-supplement in $f(M)$.

**Corollary 2.8.** Any factor module of a cofinitely weak Rad-supplemented module is a cofinitely weak Rad-supplemented module.

**3. AMPLY COFINITELY WEAK RAD-SUPPLEMENTED MODULES**

In this section, we show the concept of amply cofinitely weak Rad-supplemented modules and give some properties of such type of modules.

**Definition 3.1.** Let $M$ be an $R$-module. If every cofinite submodule of $M$ has ample weak Rad-supplements in $M$ then $M$ is called amply cofinitely weak Rad-supplemented module.

**Proposition 3.2.** Every factor module of an amply cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

**Proof:** Let $M$ be an amply cofinitely weak Rad-supplemented module. For $A \subseteq X \subseteq M$, let $M / A$ be any factor module of $M$ and $X / A$ be a cofinite submodule of $M / A$, then $M / A \cong X / A$ is finitely generated. Now $M / X \cong M / A \cong X / A$.

$M / X$ is also finitely generated. Hence $X$ is a cofinite submodule of $M$. Suppose $X / A + Y / A = M / A$ for some submodule $Y / A$ of $M / A$, then $X + Y = M$. Since $X$ is cofinite and $M$ is amply cofinitely weak Rad-supplemented, there is a submodule $B$ of $Y$ such that $B$ is a weak Rad-supplement of $X$ in $M$. By lemma 2.2, $(B + A) / A$ is a weak Rad-supplement of $X / A$ in $M / A$. Clearly $(B + A) / A \subseteq Y / A$. Hence $M / A$ is amply cofinitely weak Rad-supplemented.

**Corollary 3.3.** Every homomorphic image of an amply cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

**Proof:** Let $M$ be an amply cofinitely weak Rad-supplemented module. Since every homomorphic image of $M$ is isomorphic to a factor module of $M$, then by proposition 3.2, every homomorphic image of $M$ is amply cofinitely weak Rad-supplemented.

The $R$-module $M$ is called $\pi$-projective, if for every submodules $U$ and $V$ with $M = U + V$ there exists a homomorphism $f : M \to M$ such that $\text{Im } f \subseteq U$ and $\text{Im } (1 - f) \subseteq V$ [see, 2].

**Proposition 3.4.** Let $M$ be a cofinitely weak Rad-supplemented and $\pi$-projective module. Then $M$ is amply cofinitely weak Rad-supplemented.

**Proof:** Let $A$ be a cofinite submodule of $M$ and $A + B = M$ for any submodule $B$ of $M$. Since $M$ is cofinitely weakly Rad-supplemented and $A$ is a cofinite submodule of $M$, there exists a weak Rad-supplement $T$ of $A$ in $M$. Since $M$ is $\pi$-projective, there exists an homomorphism $f : M \to M$ such that $\text{Im } f \subseteq B$ and $\text{Im } (1 - f) \subseteq A$. Then we can show $f(A) \subseteq A$ and $(1 - f)(B) \subseteq B$. In this case

$$M = f(M) + (1 - f)(M) = A + f(A + T) = A + f(A) + f(T) = A + f(T).$$
Let \( a \in A + f(T) \). Then there exists \( t \in T \) with \( a = f(t) \). This case \( t - a = t - f(t) = (1 - f)(t) \in A \) and then \( t \in A \). Hence \( t \in A \cap T \) and \( A \cap f(T) \subseteq f(A \cap T) \). By the Hypothesis, since \( A \cap T \subseteq \text{Rad}M \) so \( f(A \cap T) \subseteq f(\text{Rad}M) \). \( A \cap f(T) \subseteq f(A \cap T) \subseteq f(\text{Rad}M) \subseteq \text{Rad}(f(M)) \subseteq \text{Rad}M \). Hence \( f(T) \) is a weak Rad-supplement of \( A \) in \( M \). Since \( f(T) \subseteq B, A \) has ample weak Rad-supplements in \( M \). Thus \( M \) is amply cofinitely weak Rad-supplemented.

**Corollary 3.5.** Every projective and cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

**Proof:** We can show that every projective module is \( \pi \)-projective module. Now by the proposition 3.4, every projective and cofinitely weak Rad-supplemented module is amply cofinitely weak Rad-supplemented.

**Lemma 3.6.** Let \( M \) be an \( R \)-module with small radical and \( A \subseteq M \). If \( A \) has a weak Rad-supplement that is a supplement in \( M \), then \( A \) has a supplement in \( M \).

**Proof:** Let \( B \) be a weak Rad-supplement of \( A \) in \( M \), then \( A \cap B \subseteq \text{Rad}M \) and so \( A \cap B \ll M \). Since \( B \) is a supplement in \( M \), \( A \cap B \ll B \). Hence \( B \) is a supplement of \( A \) in \( M \).

**Theorem 3.7.** Let \( M \) be an \( R \)-module with small radical. If \( M \) is amply cofinitely weak Rad-supplemented such that weak Rad-supplements are supplements in \( M \), then \( M \) is amply cofinitely supplemented.

**Proof:** For proof see lemma 3.6.

**Corollary 3.8** Let \( R \) be any ring. If the \( R \)-module \( R \) is weak Rad-supplemented such that weak Rad-supplements are supplements in \( R \), then \( R \) is semiperfect.

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**REFERENCES**


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