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SIMULATION RESULTS FOR WAVELET APPROXIMATION

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ABSTRACT

We investigate expansions of three functions with respect to 5 wavelet bases. Direct and inverse theorems for wavelet approximation in C and L norms are proved. For the functions possessing local regularity we study the rate of point wise convergence of wavelet Fourier series. We also define and investigate the "Discrete Wavelet Fourier transform" (DWFT) for periodic wavelets generated by a compactly supported scaling function. The DWFT has one important advantage for numerical problems compared with the corresponding wavelet Fourier coefficients: while fast computational algorithms for wavelet Fourier coefficients are recursive, DWFTs can be computed by explicit formulas without any recursion and the computation is fast enough.

The wavelet experiments in the function approximation uses five wavelets like Haar (Haar) wavelet, Debauches (db) wavelet, Symlets (sym) wavelet, Coiflet (coif) wavelet and Biorthogonal (bio) wavelet for the three functions Continuous Exponential Function (case-1), Continuous Periodic Function (Case -2) and Piecewise continuous function (Case -3).All the approximations are carried out for the data length of 2000 data points up to the level 5. The details and approximation are given below.

Key words: *Haar wavelet, Debauches wavelet, Symlets wavelet, Coiflet wavelet and Biorthogonal wavelet.*

1.1 INTRODUCTION:

The history of wavelets can be traced to many ideas developed in pure and applied mathematics, Physics and Engineering. Way back in 1910, the mathematician Alford Haar was the first to produce a complete orthonormal set for the Hilbert bert Space$L2(R)$ the elements of which are the in a sense building blocks of wavelet theory. However the interest in the field activated only during early 1980's, begins with the work of J. Morlet (1982). The results obtained by him though encouraging were not well received by the mathematical community. It was A. Grossman (1984) who laid a firm foundation to the theory. His work besides gaining mathematical respectability triggered active research in the field. The main breakthrough came only in the late 1980's with an axiomatic treatment of Multiresolution analysis by Mallat and Meyer (1986) and the method of construction of orthonormal wavelets having compact support by Ingrid Daubeachies (1987). It is because of the contributions made by these scholars and many others wavelets theory stands today as a discipline in its own right sharing borders with scientific computing, signal and image processing, Data compression (to name a few). It has been one of the major research domains in science and Engineering in the last decade and is still undergoing rapid growth.

1.2 SOME BRIEF MATHEMATICAL PRELIMINARIES:

The discrete wavelet transform decomposes a function as a sum of basis functions called wavelets. These basis functions have the property that they can be obtained by dilating and translating two basic types of wavelets known as the *scaling function*, or *father wavelet* ϕ, and the *mother wavelet* Ψ. These translations and dilations are defined as follows:

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$
$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

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The index j defines the dilation or *level* while the index k defines the translate. Loosely speaking, sums of the $\phi_{j,k}(x)$ capture low frequencies and sums of the $\psi_{j,k}(x)$ represent high frequencies in the data. More precisely, for any suitable function $f(x)$ and for any j_0,

$$f(x) = \sum_k c_k^{j_0} \phi_{j_0,k}(x) + \sum_{j \geq j_0} \sum_k d_k^j \psi_{j,k}(x)$$

where the c_k^j and d_k^j are known as the scaling coefficients and the detail coefficients, respectively. For orthonormal wavelet families these coefficients can be computed by

$$c_k^j = \int f(x) \phi_{j,k}(x) dx$$

$$d_k^j = \int f(x) \psi_{j,k}(x) dx$$

The key to obtaining fast numerical algorithms for computing the detail and scaling coefficients for a given function $f(x)$ is that there are simple recurrence relationships that enable you to compute the coefficients at level $j-1$ from the values of the scaling coefficients at level j. These formulas are

$$c_k^{j-1} = \sum_i h_{i-2k} c_i^j$$

$$d_k^{j-1} = \sum_i g_{i-2k} c_i^j$$

The coefficients h_k and g_k that appear in these formulas are called *filter coefficients*. The h_k are determined by the father wavelet and they form a low-pass filter; $g_k = (-1)^k h_{1-k}$ and form a high-pass filter. The preceding sums are formally over the entire (infinite) range of integers. However, for wavelets that are zero except on a finite interval, only finitely many of the filter coefficients are nonzero, and so in this case the sums in the recurrence relationships for the detail and scaling coefficients are finite.

Conversely, if you know the detail and scaling coefficients at level $j-1$, then you can obtain the scaling coefficients at level j by using the relationship

$$c_k^j = \sum_i h_{k-2i} c_i^{j-1} + \sum_i g_{k-2i} d_i^{j-1}$$

Suppose that you have data values

$$y_k = f(x_k), \qquad k = 0, 1, 2, \cdots, N-1$$

at $N = 2^J$ equally spaced points x_k. It turns out that the values $2^{-J/2} y_k$ are good approximations of the scaling coefficients c_k^J. Then, by using the recurrence formula, you can find c_k^{J-1} and d_k^{J-1}, $k = 0, 1, 2, \cdots, N/2-1$. The discrete wavelet transform of the y_k at level $J-1$ consists of the $N/2$ scaling and $N/2$ detail coefficients at level $J-1$ A technical point that arises is that in applying the recurrence relationships to finite data, a few values of the c_k^J for $k < 0$ or $k \geq N$ might be needed. One way to cope with this difficulty is to extend the sequence c_k^J to the left and right by using some specified boundary treatment.

Continuing by replacing the scaling coefficients at any level j by the scaling and detail coefficients at level $j-1$ yields a sequence of N coefficients

$$\{c_0^0, d_0^0, d_0^1, d_1^1, d_0^2, d_1^2, d_2^2, d_3^2, d_1^3, \ldots, d_7^3, \ldots, d_0^{J-1}, \ldots, d_{N/2-1}^{J-1}\}$$

This sequence is the finite discrete wavelet transform of the input data $\{y_k\}$. At any level $j_0$the finite dimensional approximation of the function $f(x)$is

$$f(x) \approx \sum_k c_k^{j_0} \phi_{j_0,k}(x) + \sum_{j=j_0}^{J-1} \sum_k d_k^j \psi_{j,k}(x)$$

(A) CONTINUOUS EXPONENTIAL FUNCTION(CASE1):

(i) Haar:

Wavelet decomposition and reconstruction have been carried out for the continuous Exponential function. The approximation and details up to the level 5are given figure 1(a) the approximation and the Statistical are given in figure 1(b)

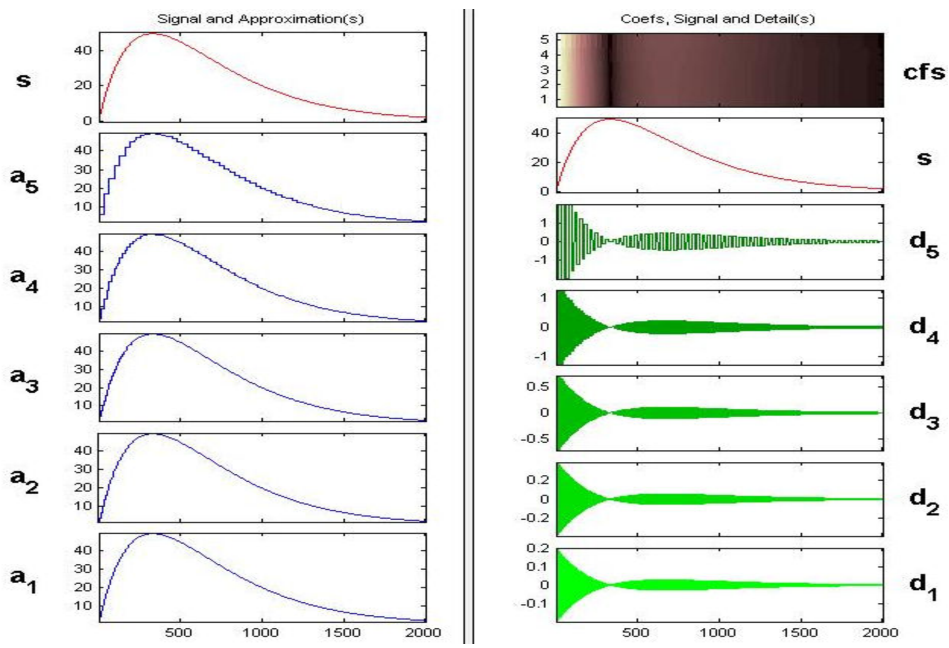


Fig 1 (a) Continuous exponential function by Haar wavelet

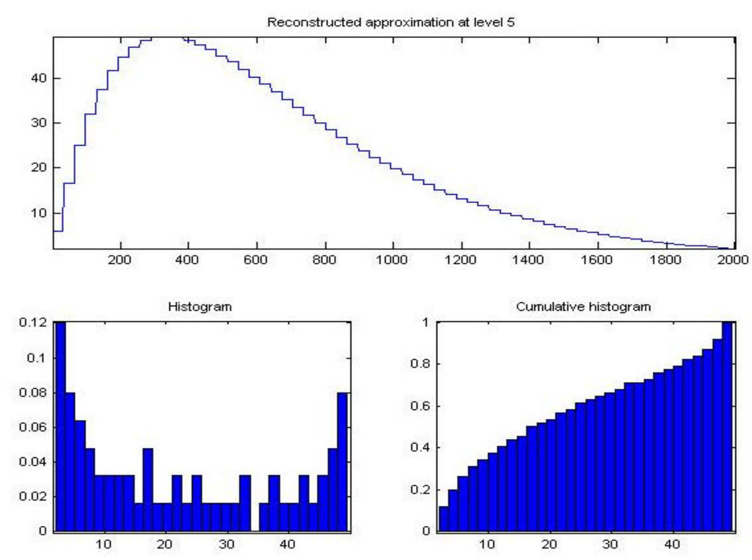


Fig1 (b) Statistics of continuous exponential function by Haar wavelet

In the haar wavelet experiment (Figure-1) approximations up to the level of 5 depicts successive function approximation with noise removed. The detailed coefficients show successive corresponding noise components. Exponential function approximations of the level 3 shows smoothness were as the approximation level 4,5show discrete unsmooth features. This may be the result of over estimation beyond the level of 3 .However the plotted the reconstruction level shows commutative smoothness of the entire range of the function. The characteristic of noise path remains low up to the level 3 with slight increase at d4 and d5. The approximation of error computed from the norm RMSE=4.5092e-016.It shows the haar wavelet at level 5 has minimal error therefore the R-Square Value is =1.

(ii) Db4:

The Db4 is uniformly smooth across the approximation. The cumulative Histogram and the level of 5 shows uniform characteristics of the Histogram as expected of an exponential function. In keeping with the nature of experiments with the RMSE is found to be 4.6624e-014. With such minimum error at level 5 approximation the R-Square Value found to be 1.

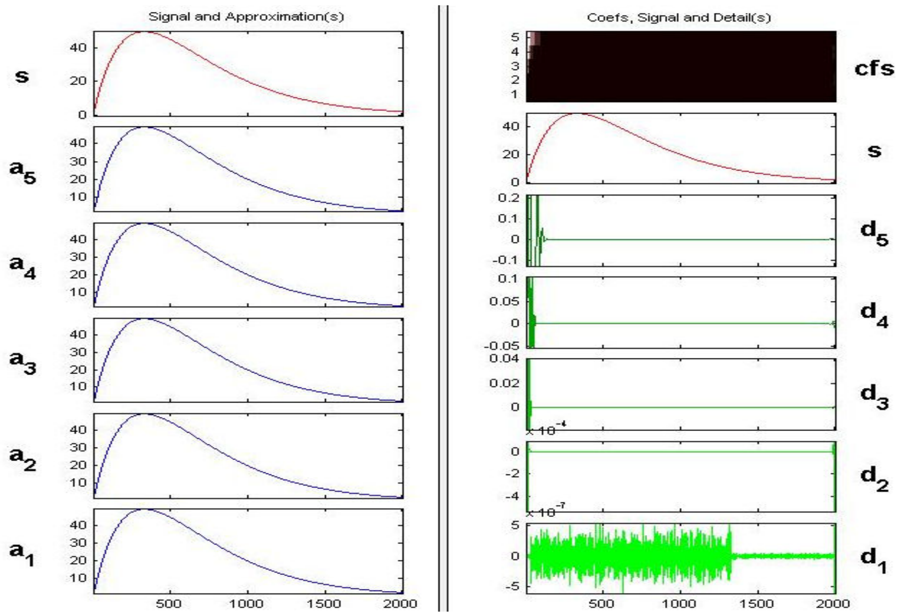


Fig 2 (a) Continuous exponential function by db 4

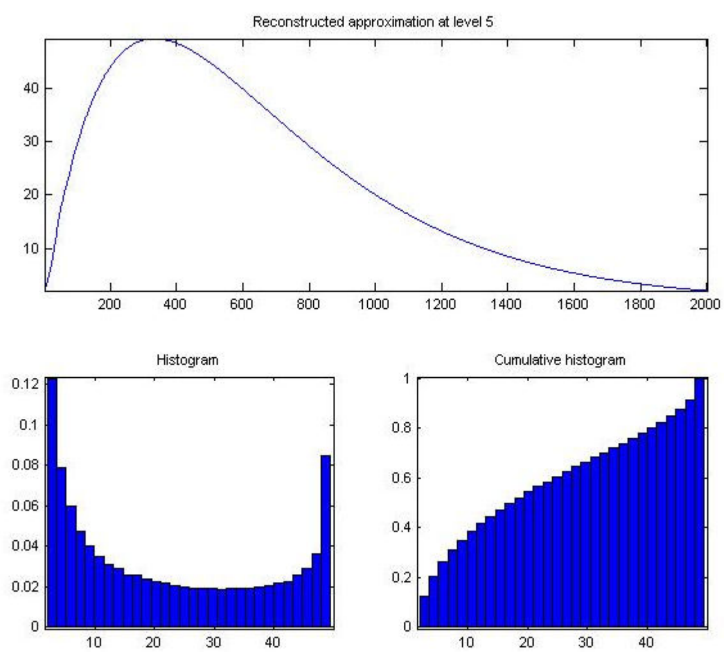


Fig2 (b) Statistics of Continuous exponential function by db 4

(iii) Symlets:

In this symlets up to the level of 5 for Exponential function approximation the shows smoothness at all levels. We have plotted level with its Histogram and cumulative Histogram of the approximation. The approximation of error RMSE = 4.8542e-015 and R-Square is 1.

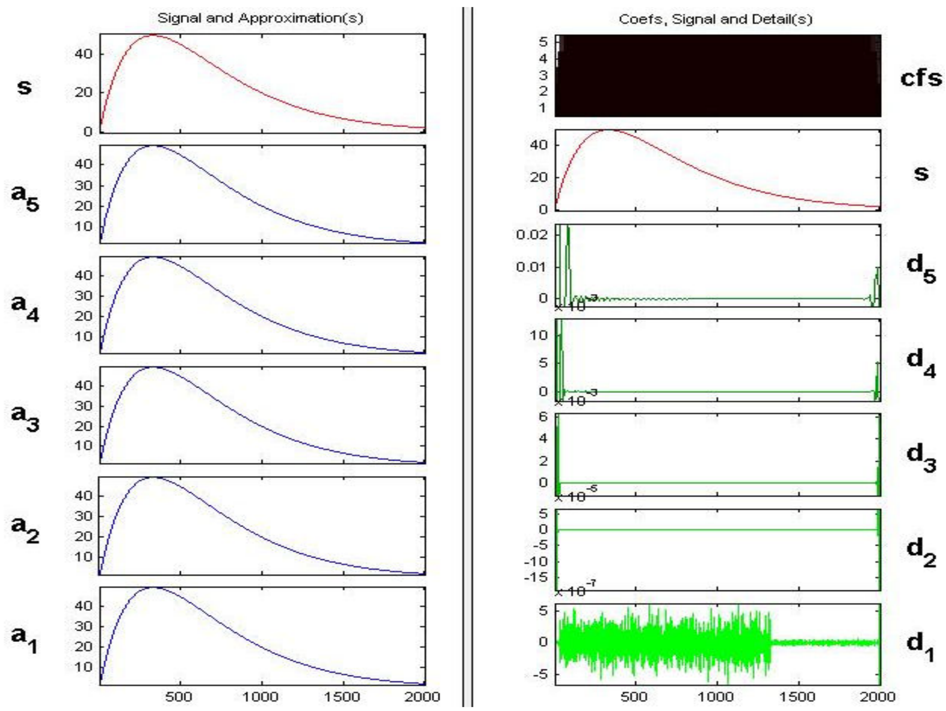
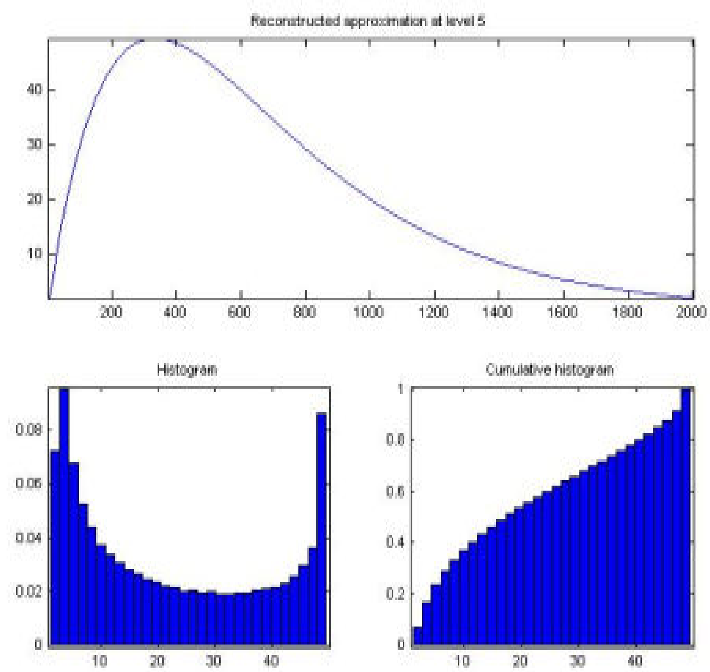


Fig 3 (a) Continuous exponential function by sym 5



3 (b) Statistics of Continuous exponential function by sym 5

(iv) Coiflets:

The coiflets is uniformly smooth across the approximations. The histogram and cumulative Histogram and the level of 5 shows uniformity as expected of continuous exponential function. In keeping with the nature of experiments the evaluated RMSE is 1.6384e-010 and R-Square is 1.

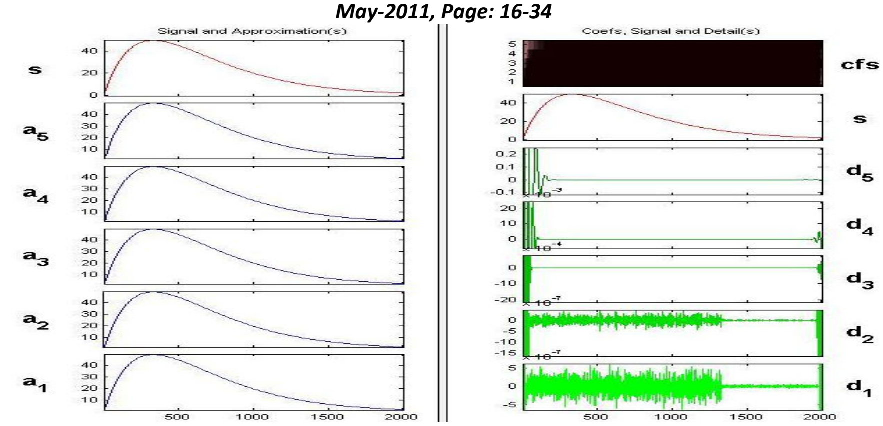


Fig 4 (a) Continuous exponential function by coif 5

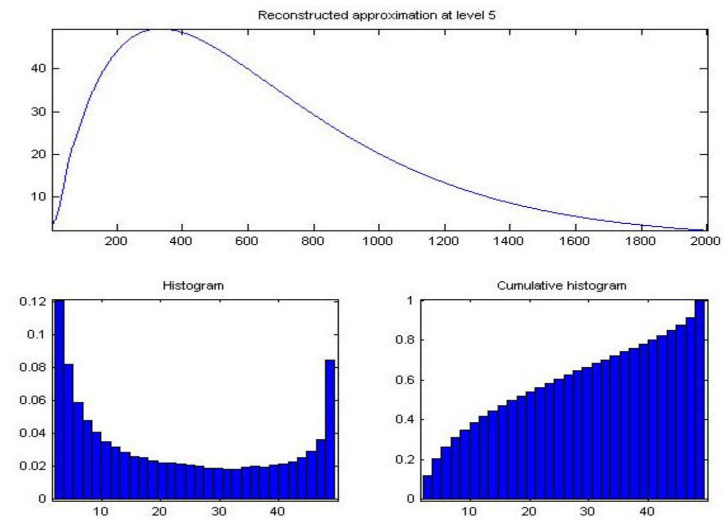


Fig.4 (b) Statistics Continuous exponential function by coif 5

(v) Bi-orthogonal (3.3):

The figure 5(a) and 5(b) shows wavelet decomposition, reconstruction of approximation and details using bi-orthogonal (3.3) up to the level of 5 . The Exponential function approximation shows smoothness and the level 3, 4,5. The noise component in d1 and d2 are higher than level 3, 4, and we have plotted 5 level approximation Histogram and cumulative Histogram. The RMSE is found to be 4.8283e-016 and R-Square is 1.

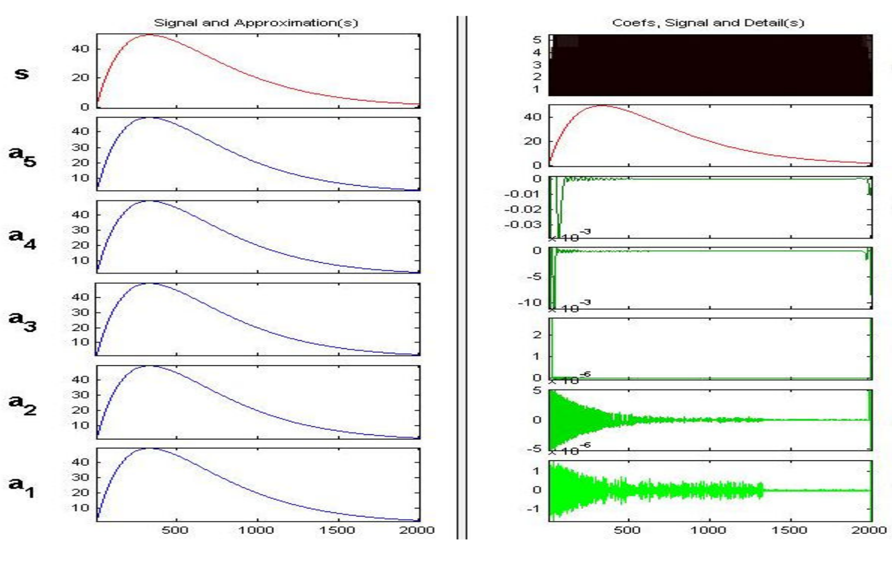


Fig 5 (a) Continuous exponential function by Bior 3.3

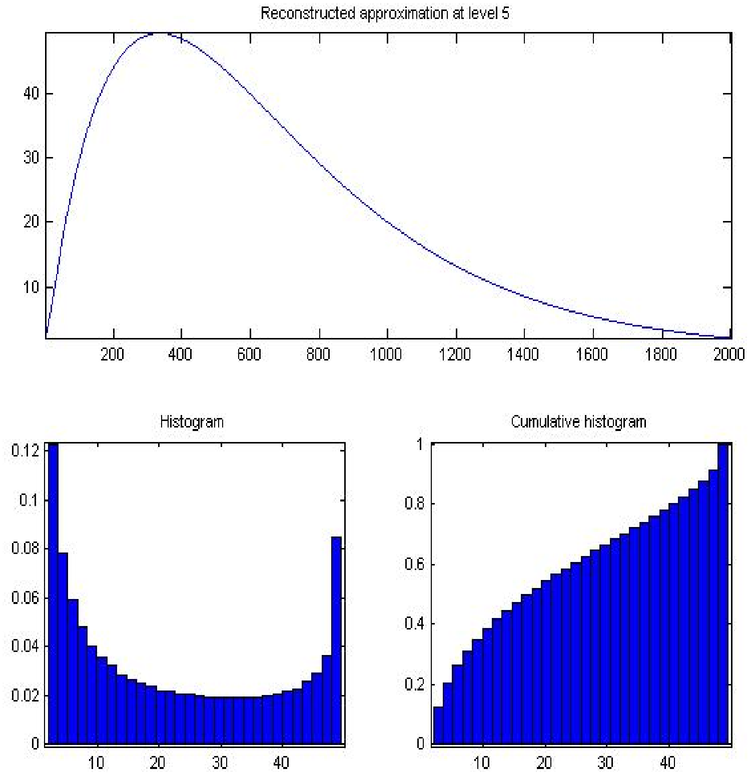


Fig 5. (b) Statistics Continuous exponential function by Bior 3.3

The signal statistics is given below Table no1 .the comparative table of RMSE's of all functions for the five wavelets are given in Table no.2

Table 1 Statistics Continuous exponential function by using wavelets

Mean	22.02	Maximum	49.47	Standard dev.	16.37	L1 norm	4.405e+004
Median	18.04	Minimum	2.017	Median abs. dev.	13.45	L2 norm	1227
Mode	2.808	Range	47.45	Mean abs. dev.	14.6	Max norm	49.47

Table 2 Error in Continuous exponential function by using wavelets

Name of the Wavelet	Haar wavelet	db wavelet	Symlet wavelet	Coiflet wavelet	Biorthogonal wavelet
RMSE	4.5092e-016	4.6624e-014	4.8542e-015	1.6384e-010	4.8283e-016

(b) Continuous Periodic Function (Case 2):

(i) Haar:

In the haar wavelet experiment (Figure 9.21) approximations up to the level of 5 successive function approximation with noise removed. The detailed coefficients show successive corresponding noise components. Continuous periodic function approximations of the level 3 shows smoothness where as the approximation level 4, 5 show discrete unsmooth features. This may be the result of over estimation beyond the level of 3 .However the plotted reconstruction level shows commutative smoothness of the entire range of the function. The approximation of error computed from the norm RMSE=3.3559e-018.It shows the haar wavelet at level 5 has minimal error therefore the R-Square Value is =1.It is uniformly smooth across the approximation. The histogram is not correlated in every point and cumulative Histogram at level of 5 shows characteristics of nonlinearity function.

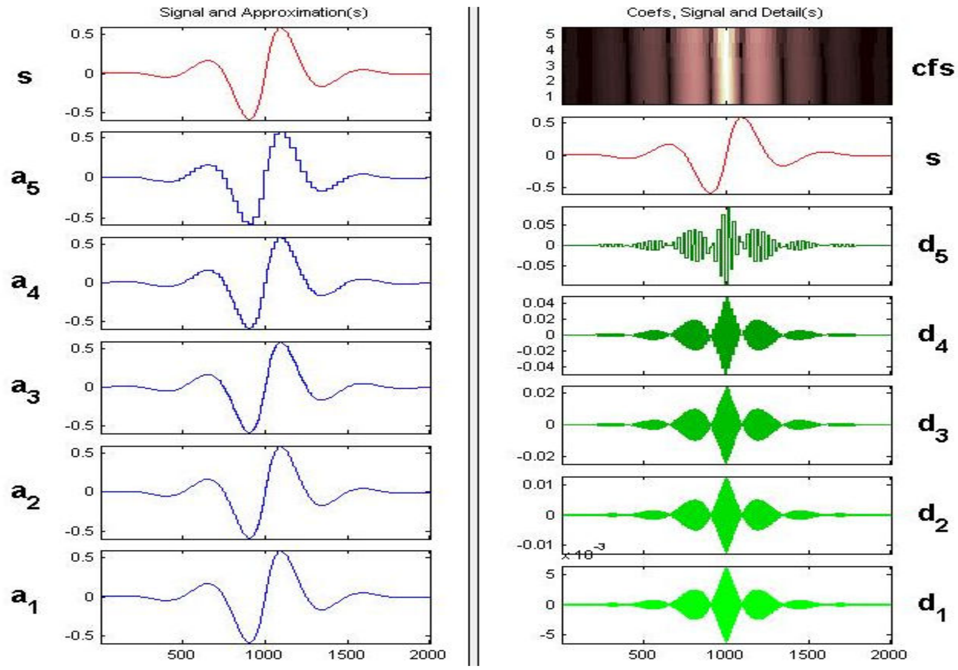


Fig 6 (a) Continuous periodic functions by Haar

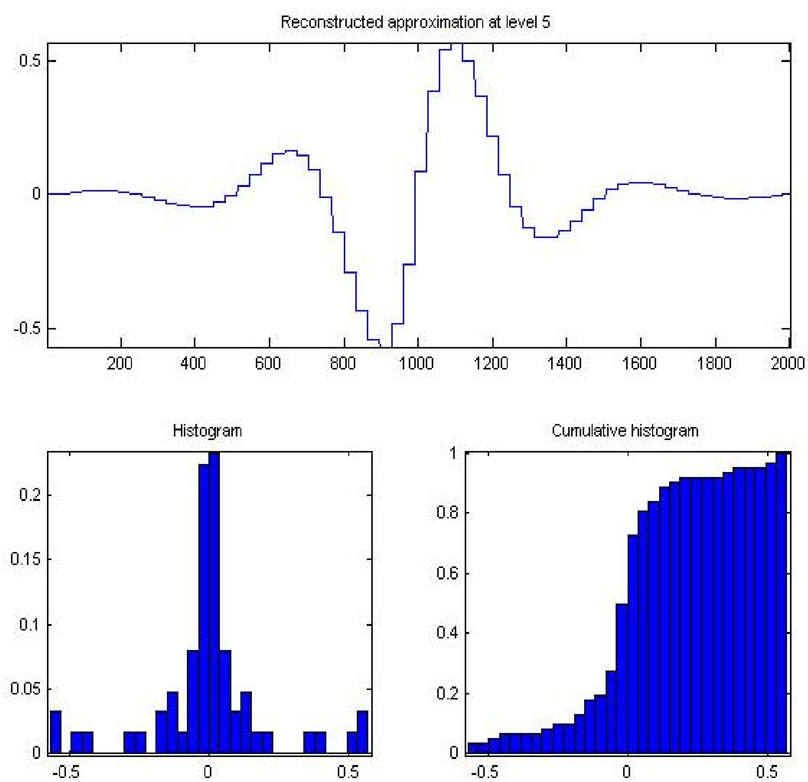


Fig.6 (b) Statistics Continuous periodic function by Haar

(ii)Db4:

The Db4 is uniformly smooth across the approximation. The cumulative Histogram and the level of 5 shows uniform characteristics of the Histogram as expected of an continuous periodic function. In keeping with the nature of experiments with the RMSE is found to be 2.7361e-015. With such minimum error at level 5 approximation the R-Square Value found to be 1.Unlike Harr, the Db4 is uniformly smooth across the approximation noise is seen only at the transitional zone. Therefore, db4 may be considered a good approximation tool.

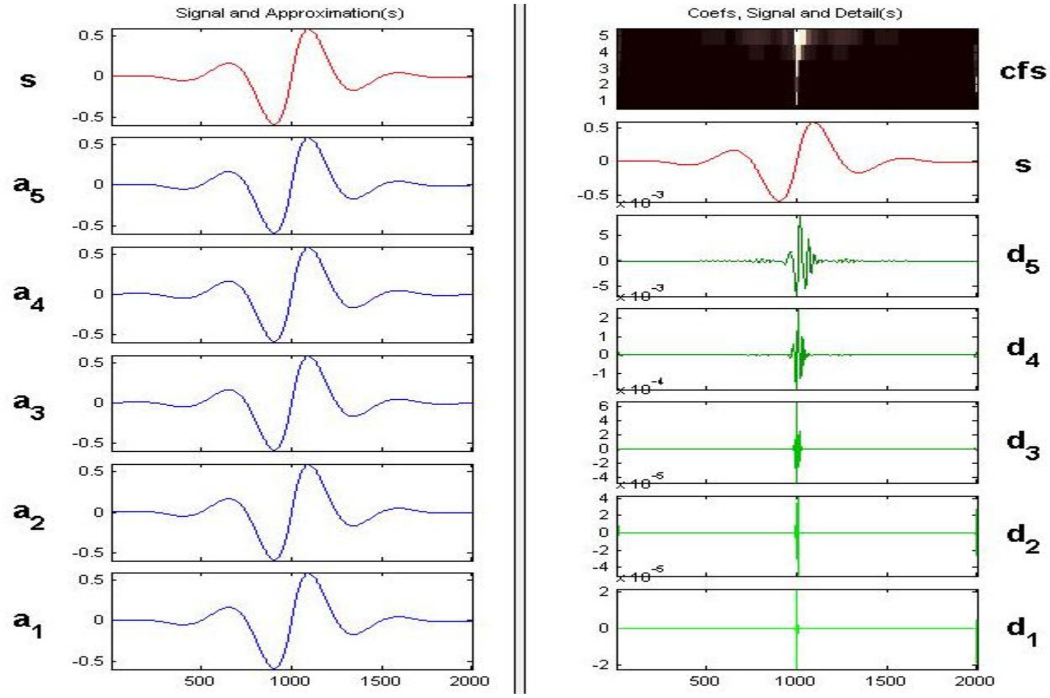


Fig 7 (a) Continuous Periodic function by db 4

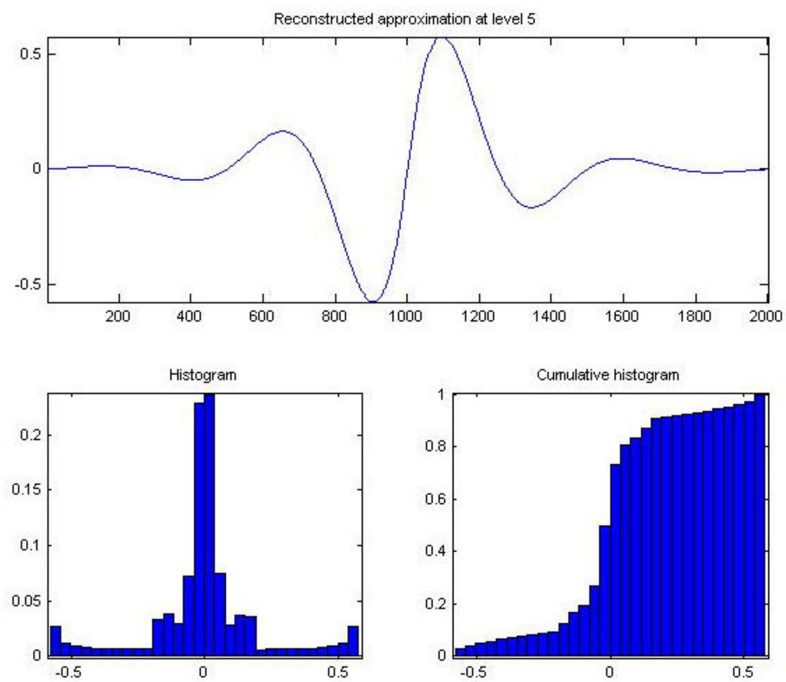


Fig 7(b) Statistics periodic exponential function by db 4

(iii) Symlets:

In this symlets up to the level of 5 for periodic continuous function approximation the shows smoothness at all levels. The results are almost similar or better than db4. We have plotted level with its Histogram and cumulative Histogram of the approximation.

In keeping with the nature of experiments with the RMSE is found to be 6.0941e-017. With such minimum error at level 5 approximation the R-Square Value found to be 1.

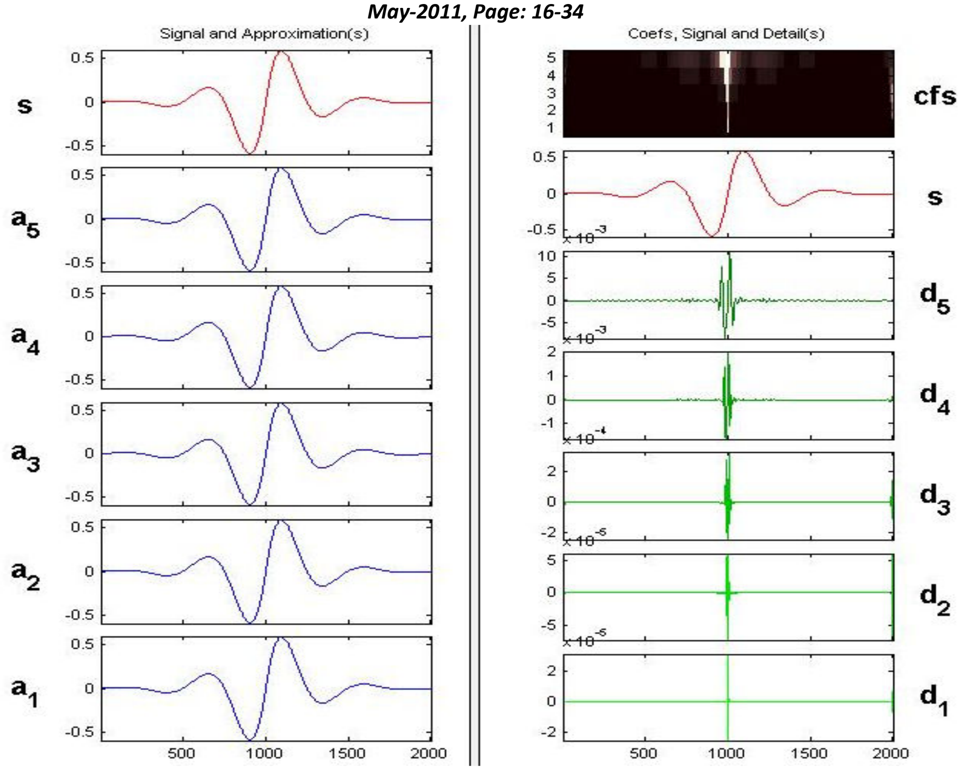


Fig 8(a) Continuous periodic function by sym4

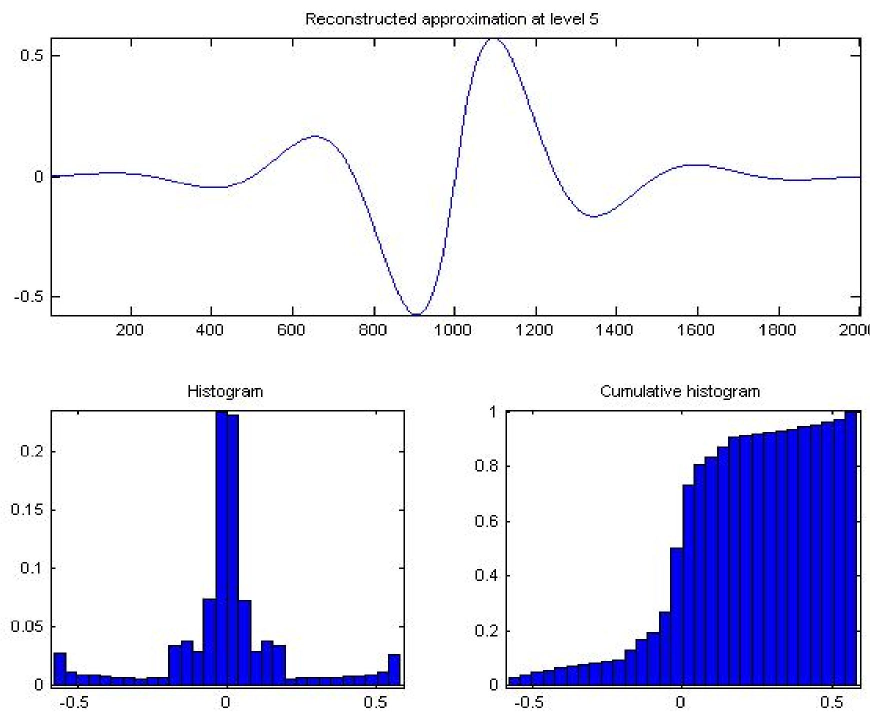


Fig 7 (b) Statistics periodic exponential function by sym 4

(iv) Coiflets:

The coiflets is uniformly smooth across the approximation for the continuous periodic function. The pattern is similar to db4 and symlet.The histogram is not correlated in every point and cumulative Histogram and the level of 5 shows uniform smooth transition characterizing the nonlinear nature of the periodic function. The evaluated RMSE is2.8961e-012and R-Square is 1.

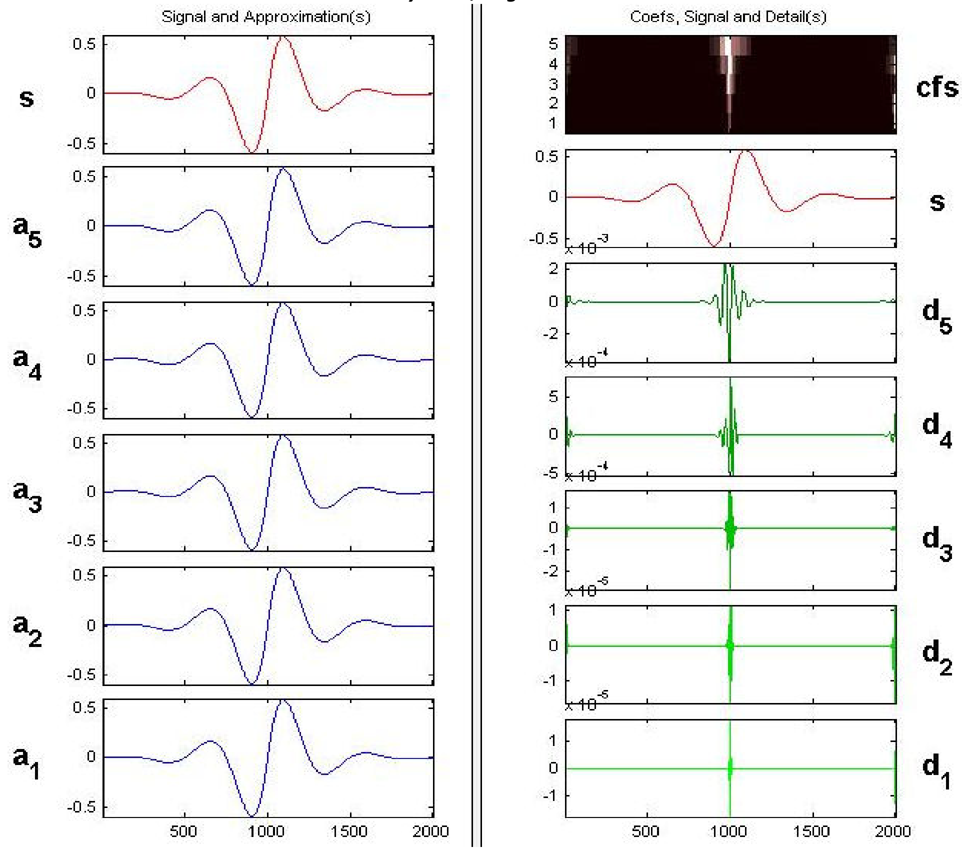


Fig 9 (a) Continuous periodic function by coif 5

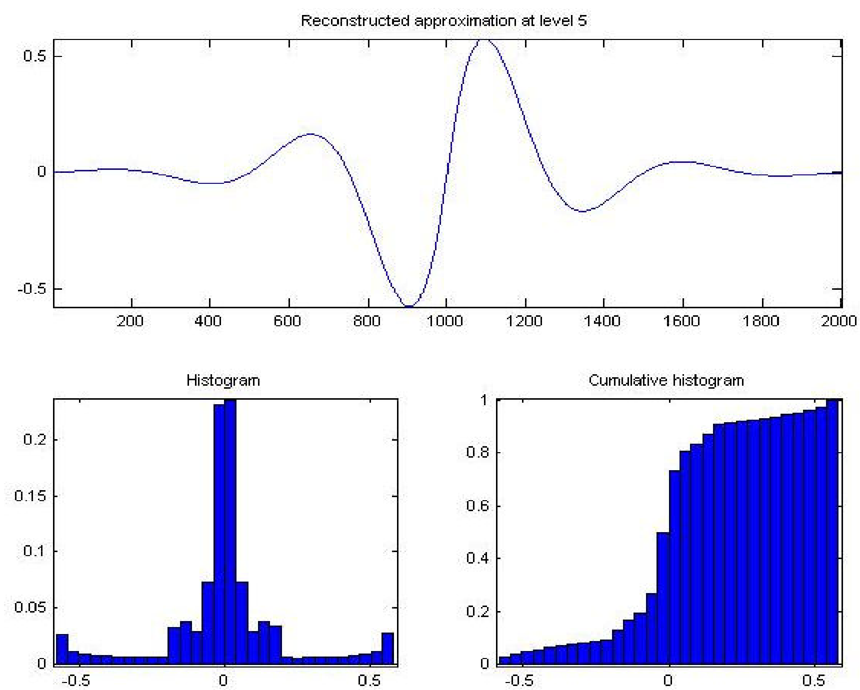


Fig 9 (b) Statistics Continuous periodic function by coif 5

(v) Bio-orthogonal (3.3):

The figure 10(a) and Fig 10(b) shows wavelet decomposition, reconstruction of approximation and details using bi-orthogonal (3.3) up to the level of 5 . The periodic function approximation shows smoothness for all the level. We have plotted 5 level approximation Histogram and cumulative Histogram. The RMSE is found to be 3.4384e-018 and R-Square is 1. The error is the least across wavelets used in the experiments. The noise component is equally limited to the transition points. Bi-orthogonal wavelet may considered as the most successful among the five wavelets in this experiment.

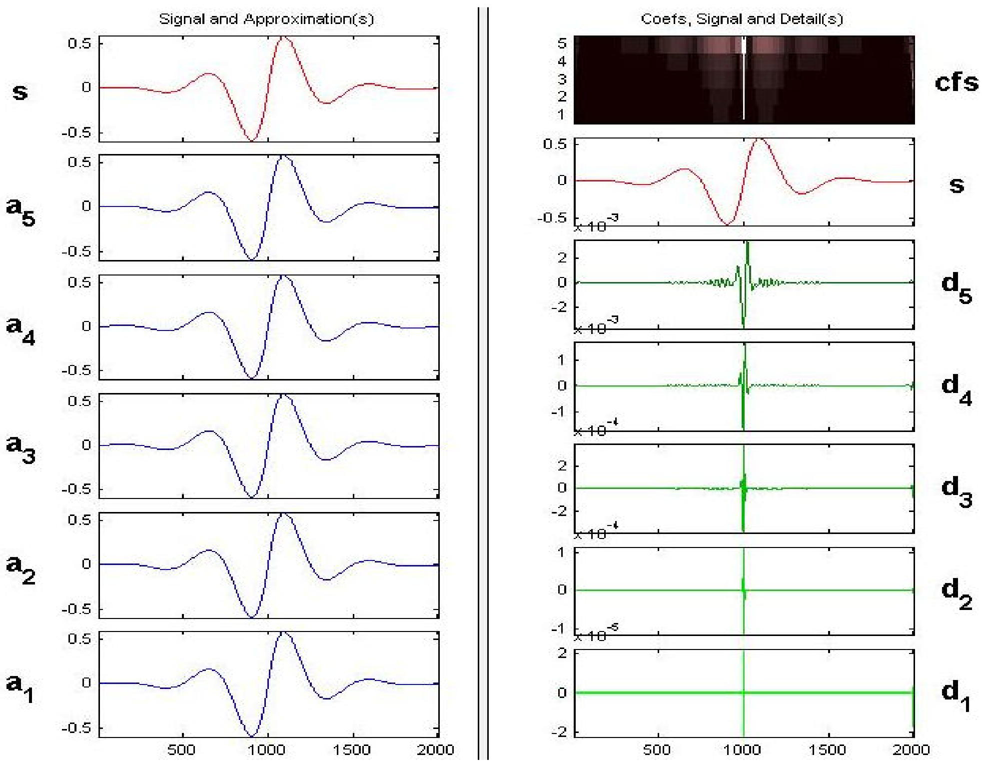


Fig 10 (a) Continuous periodic function by Bior 3.3

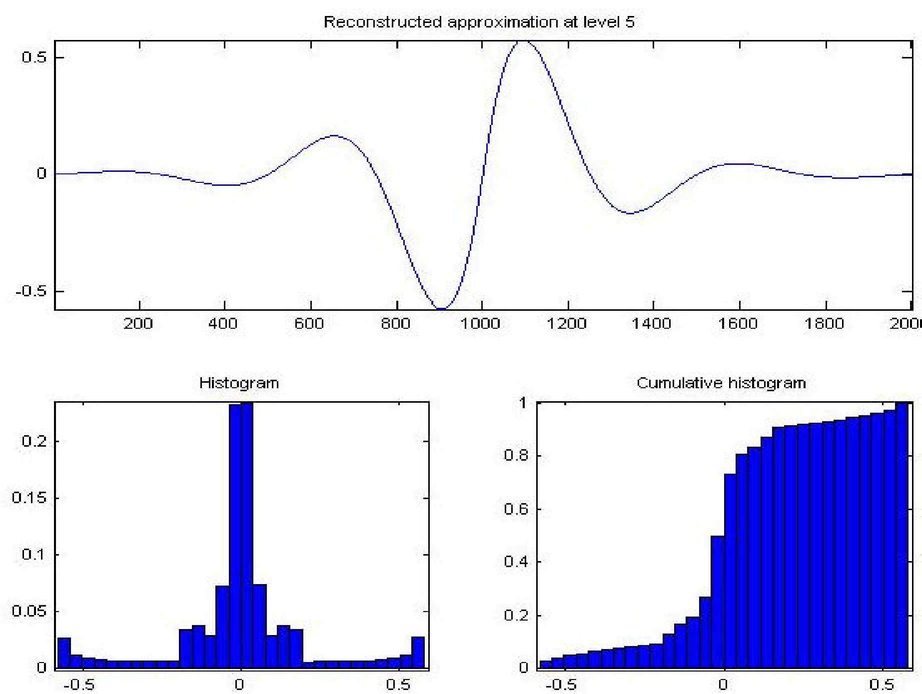


Fig. 10(b) Statistics Continuous periodic function by Bior 3.3

The signal statistics is given below Table no. 3 .the comparative table of RMSE's of all functions for the five wavelets are given in Table no.4

Table 3 Statistics Continuous periodic function by using all wavelets

Mean	-2.757e-007	Maximum	0.5779	Standard dev.	0.2078	L1 norm	246
Median	1.264e-006	Minimum	-0.5785	Median abs. dev.	0.04304	L2 norm	9.291
Mode	0.019	Range	1.156	Mean abs. dev.	0.1229	Max norm	0.5785

Table 4 Error in Continuous periodic function by using all wavelets

Name of the Wavelet	Haar wavelet	db wavelet	Symlet wavelet	Coiflet wavelet	Biorthogonal wavelet
RMSE	3.3559e-018	2.7361e-015	6.0941e-017	2.8961e-012	3.4384e-018

(c) Piecewise continuous function (Case 3):

(i) Haar:

In the haar wavelet experiment (Figure 11(a) and (b)) approximations depicts successive function approximation up to the level of 5with noise removed. The detailed coefficients show successive corresponding noise components. Unlike continuous exponential function, haar approximation is uniformly smooth across the all levels. The plotted the reconstruction level shows commutative smoothness of the entire range of the function despite higher noises in the detailed components. The characteristic of noise path remains low up to the level 2 and 3 with slight increase at d4 and d5. The approximation of error computed from the norm RMSE=2.0985e-017 It shows the haar wavelet at level 5 has minimal error therefore the R-Square Value is =1.

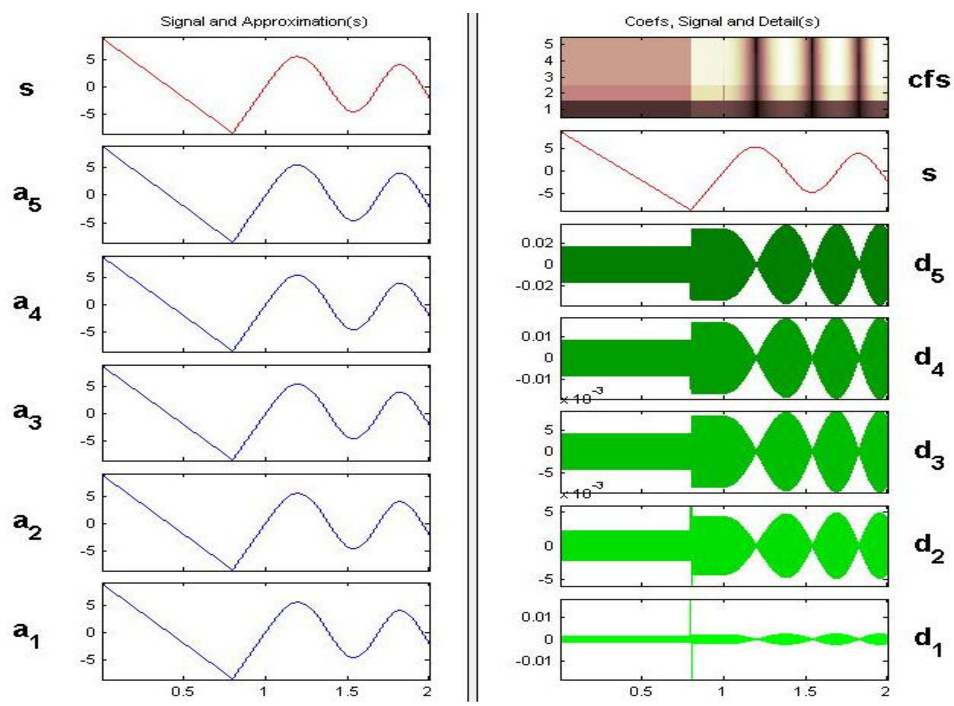
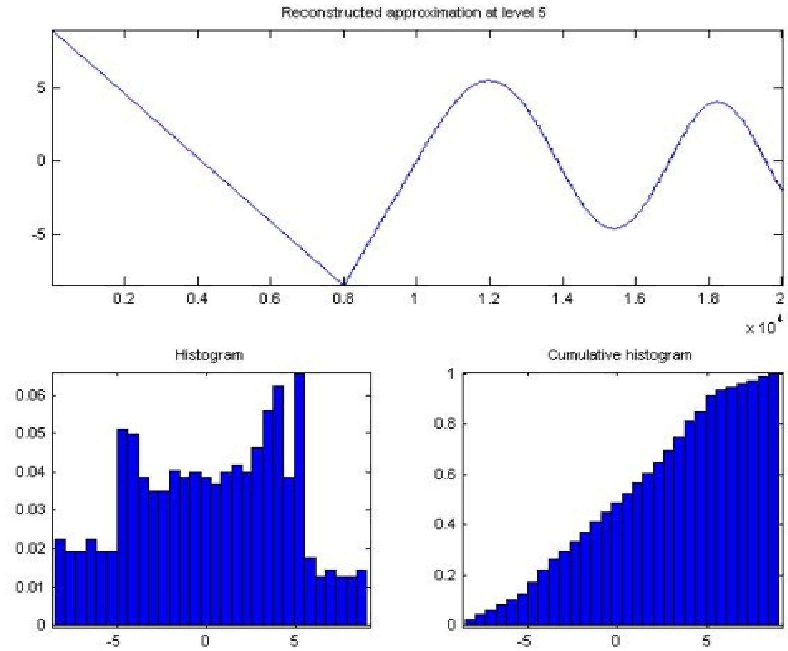


Fig 11(a) Piece wise Continuous function by Haar



11 (b) Statistics of Piece wise Continuous function by Haar

(ii) Db4:

The Db4 is uniformly smooth across the approximation. The cumulative Histogram and the level of 5 shows uniform characteristics of the Histogram as expected of an piecewise continuous function. In keeping with the nature of experiments with the RMSE is found to be 1.7179e-016. With such minimum error at level 5 approximation the R-Square Value found to be 1. While the approximation is accurate the detailed component can detect the sharp transient/ singularity of the function.

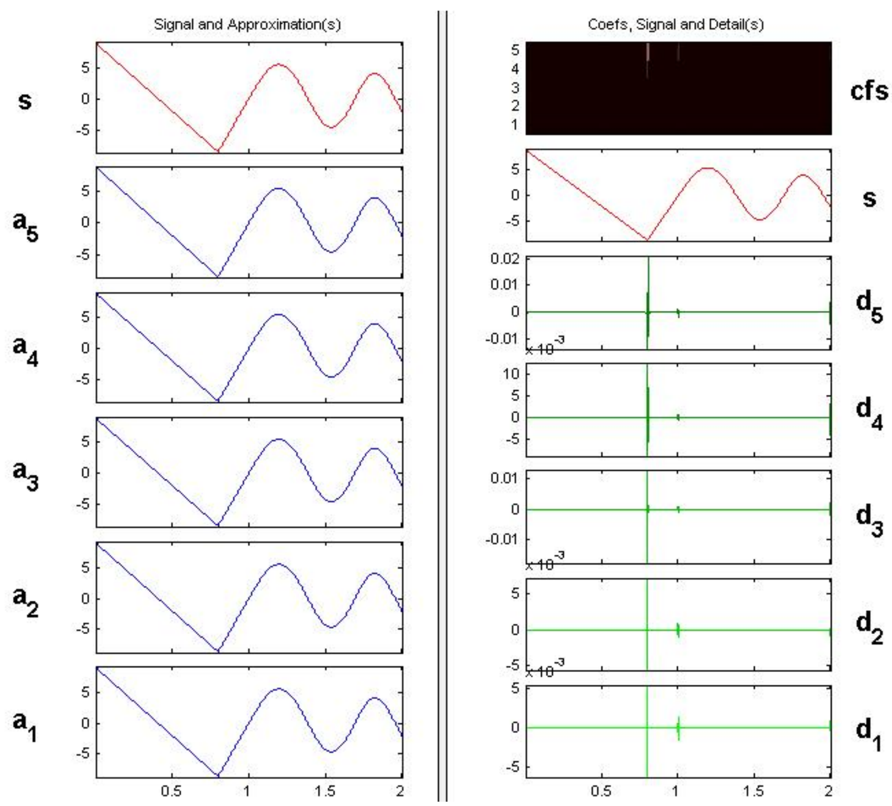


Fig 12 (a) Piece wise Continuous function by db 4

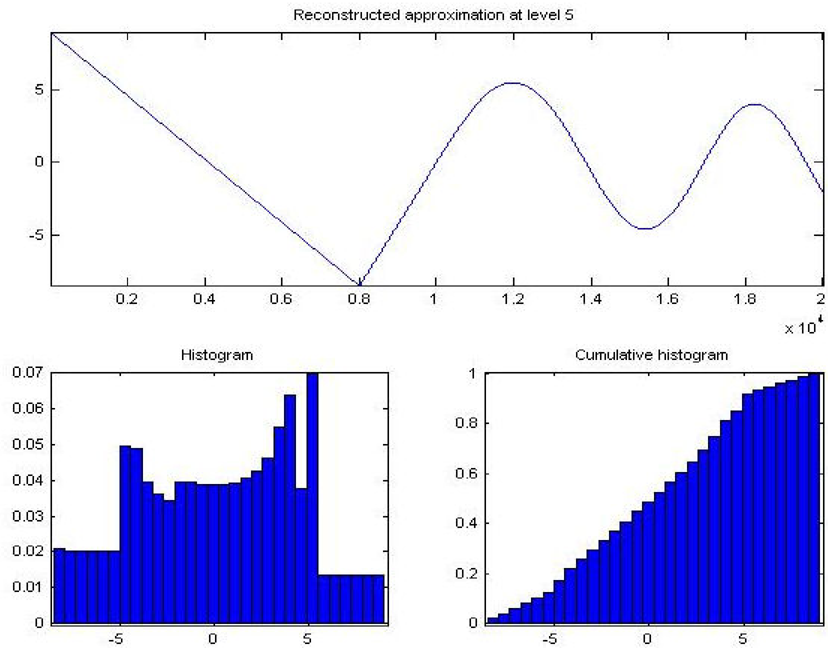


Fig12 (b) Statistics of Piece wise Continuous function by db 4

(iii) Symlets:

In this symlets up to the level of 5 for piecewise continuous function approximation the shows smoothness at all levels. The results are comparable to that of db4 ,we have plotted level with its Histogram and cumulative Histogram of the approximation. The approximation of error RMSE = 6.0827e-017and R-Square is 1.

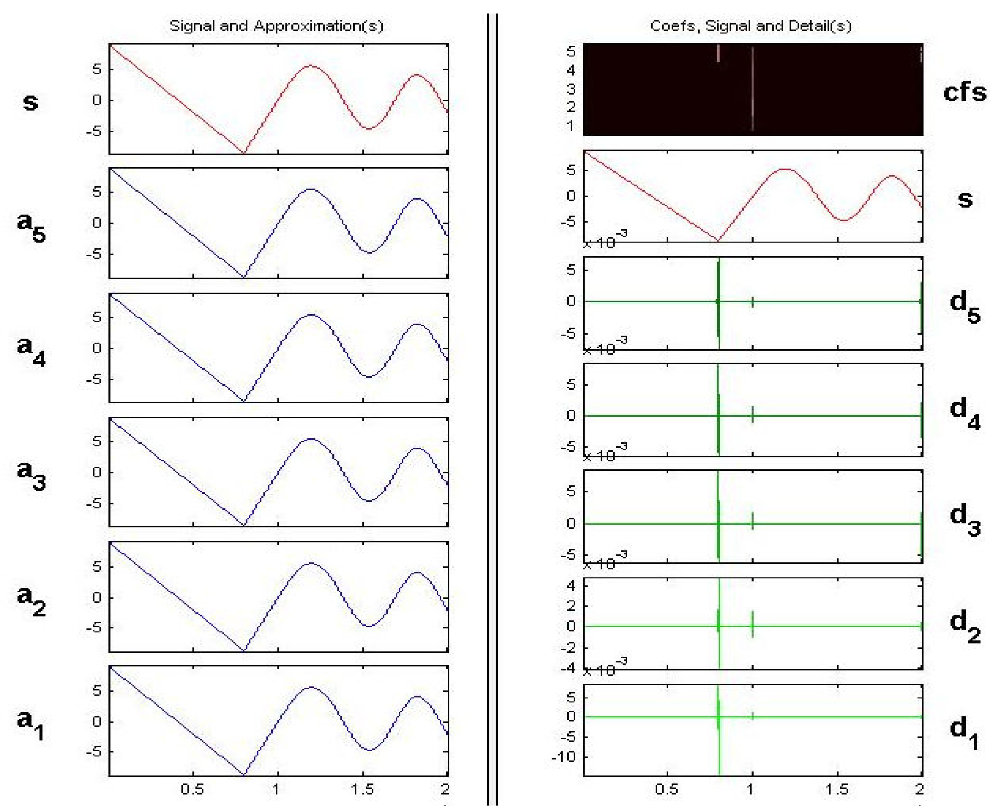


Fig 13 (a) Piece wise Continuous function by sym 4

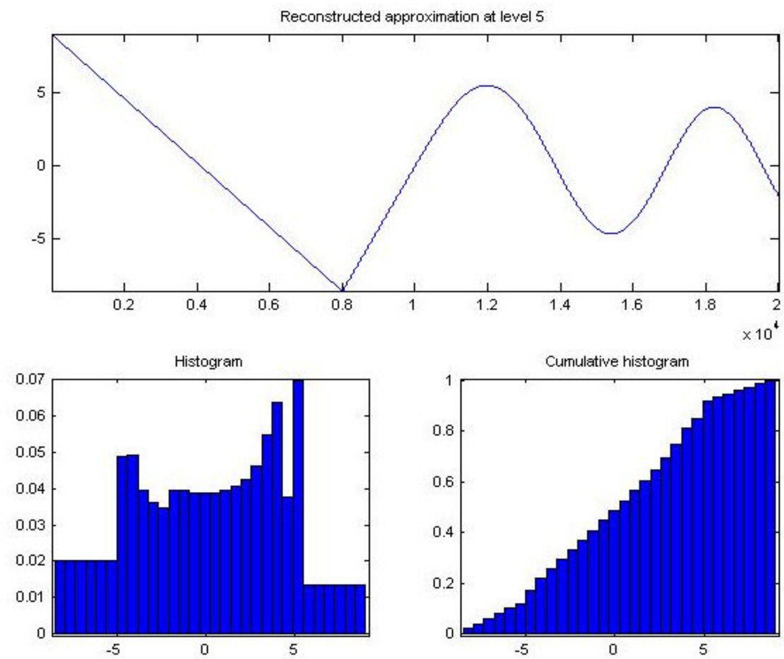


Fig13 (b) Statistics of Piece wise Continuous function by sym 4

(iv) Coiflets:

The coiflets is uniformly smooth across the approximations. The histogram and cumulative Histogram and the level of 5 shows uniformity as expected of piecewise continuous function. In keeping with the nature of experiments the evaluated RMSE is 2.0553e-013 and R-Square is 1.

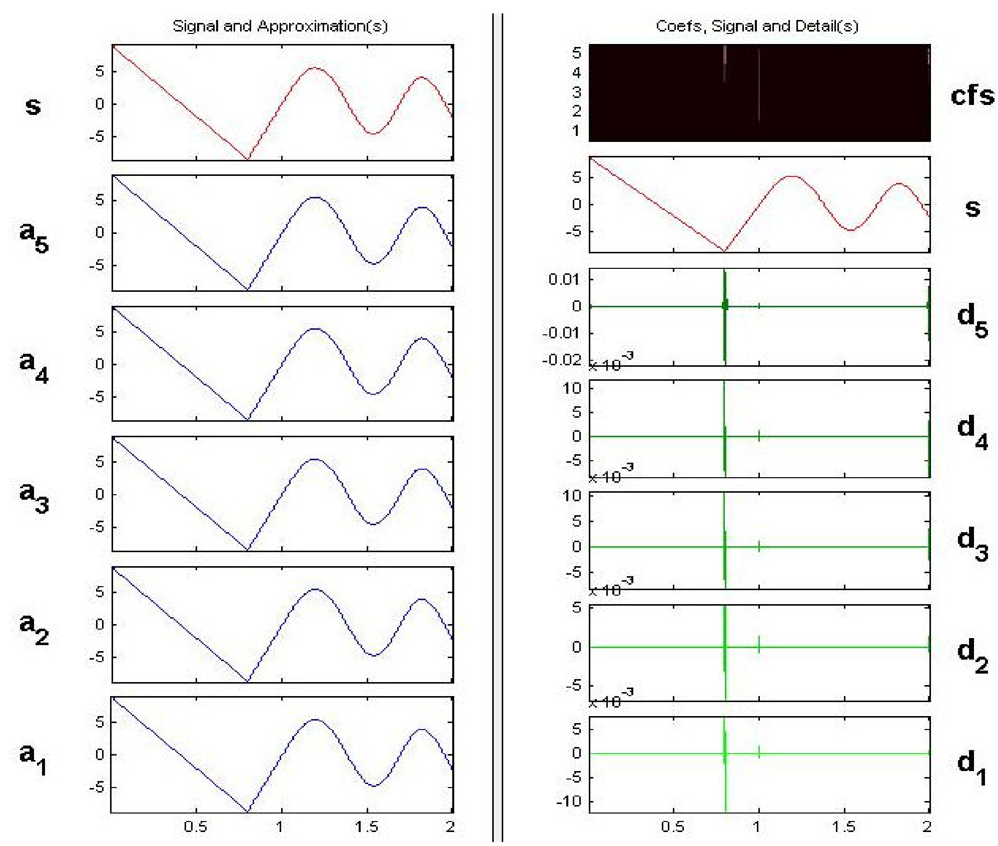


Fig 14 (a) Piece wise Continuous function by coif 5

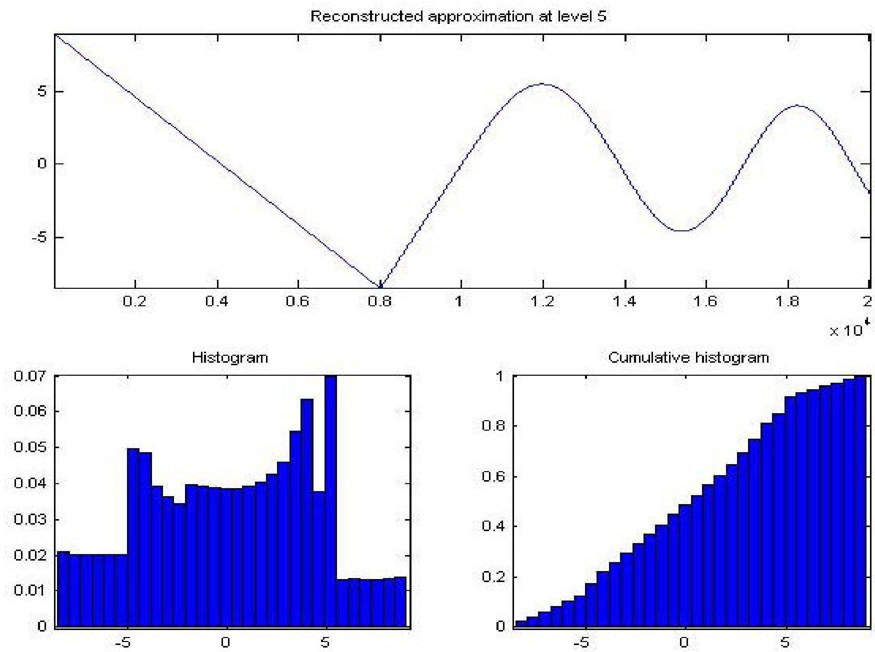


Fig14 (b) Statistics of Piece wise Continuous function by coif 5

(v) Biorthogonal (3.3):

The figure 16(a) and (b) shows wavelet decomposition, reconstruction of approximation and details using bio-orthogonal (3.3) up to the level of 5 . The piecewise continuous function approximation shows uniformly smoothness across the all the level. The histogram is not correlated in every point and cumulative Histogram and the level of 5 shows characteristics of piecewise function.

We have plotted 5 level approximation Histogram and cumulative Histogram. The RMSE is found to be 2.1714e-017and R-Square is 1. The results are comparable to that of db4, symlet and coif wavelets.

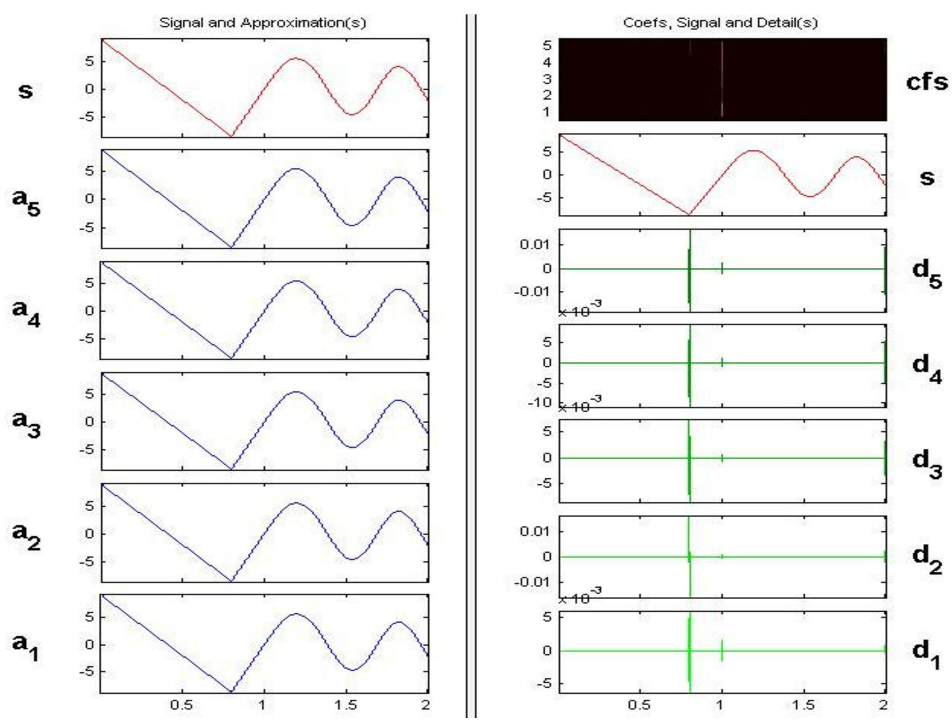
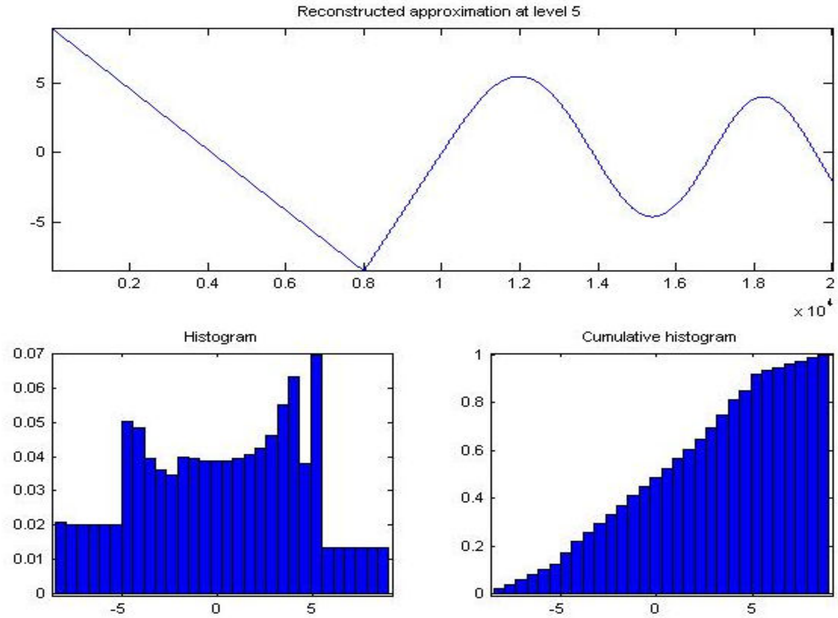


Fig 15 (a) Piece wise Continuous function by bior 3.3



(b) Statistics of Piece wise Continuous function by bior 3.3

The signal statistics is given below Table no.5 .the comparative tables of RMSE's of all functions for the five wavelets are given in Table no.6

Table 5 Statistics of Piece wise Continuous function by using wavelets

Mean	0.2065	Maximum	8.985	Standard dev.	4.269	L1 norm	7.318e+004
Median	0.4498	Minimum	-8.48	Median abs. dev.	3.481	L2 norm	604.4
Mode	5.201	Range	17.46	Mean abs. dev.	3.649	Max norm	8.985

Table 6 Error of Piece wise Continuous function by using wavelets

Name of the Wavelet	Haar wavelet	db wavelet	Symlet wavelet	Coiflet wavelet	Bi-orthogonal wavelet
RMSE	2.0985e-017	1.7179e-016	6.0827e-017	2.0553e-013	2.1714e-017

1.3 Performance Comparison of wavelet approximations:

Names of wavelets	Exponential function	Periodic function	Piecewise continuous function
Haar	4.5092e-016	3.3559e-018	2.0985e-017
db	4.6624e-014	2.7361e-015	1.7179e-016
sym	4.8542e-015	6.0941e-017	6.0827e-017
coif	1.6384e-010	2.8961e-012	2.0553e-013
bior	4.8283e-016	3.4384e-018	2.1714e-017

RESULTS AND CONCLUSION:

The results of RMSE's indicate the five wavelets can approximate all the three function carried out in these experiments. The comparative results show Haar approximation is out performs to an order for periodic function .The Haar wavelet also performs better for piecewise continuous function almost similar to bi-orthogonal wavelets. The symlets are suitable for periodic and piecewise functions. The coif wavelet is found to perform better for piecewise continuous function than the other functions .The Bi-orthogonal wavelets are most suitable for periodic function similar to Haar and approximate quit well for piecewise continuous function similar to Haar.

REFERENCE:

[1] A. Ganesh, G. Balasubramanian, S. K. Jena, N. Pradhan “Fourier Approach to Function Approximation” International Journal of Mathematical Archive Vol.-2(4), April-2011, ISSN 2229-5046 page no.-617-624

[2] A. Ganesh , G. Balasubramanian , S. K. Jena, N. Pradhan "A comparision study of Function approximation using Fourier and Wavelet Transforms" International Journal of Mathematical Archive Vol.-2(5), May-2011, ISSN 2229-5046.

[3] M. Cannone. Ondelettes, Paraproduits et Navier-Stokes. Diderot, Paris. 1995.

[4] W. K. Chen. Passive and Active Filters. John Wiley and Sons, New York, 1986.

[5] C. K. Chui. An Introduction to Wavelets. Academic Press, New York, 1992.

[6] C. K. Chui, editor. Wavelets: A Tutorial in Theory and Applications. Academic Press. New York. 1992.

[7] A. Cohen Wavelets and Multiscale Signal Processing. Chapman and Hall, 1995.

[8] bJ. M. Combes, A. Grossmann, and P.Tchamitchian, editors. Wavelets time-frequency methods and phase space. Springer-Verlag, Berlin, 1989.

[9] T. M. Cover and J. A. Thomas. Elements of Information Theory Wiley Inter- science, 1991.

[10] A. Ganesh, G. Balasubramanian “Region Based in a retrieval using wavelet transforms” Excel India Publishers, New Delhi ,India ISBN No. 93-80043-05-8 Page No.215-221.

[11] A. Ganesh, G. Balasubramanian “On Signal Processing by using Fourier Analysis of Time Series”G J P A M Online ISSN: 0973-9750 Volume 6 Number 1 (2010).

[12] A. Ganesh, G. Balasubramanian “Tuberculosis Disease Predictions and Classifications using Supervised Artificial Neural networks (ANN)” Excel India Publishers, New Delhi, India ISBN No.81-9071-964-5 Page No-24-31.

[13] A. Ganesh, G. Balasubramanian “Mathematical Modeling for Computational Simulation of the Temperature Distribution during the Synthesis of Polycrystalline Diamond” is published in the International Journal on Information Sciences and Computing, ISSN 0973-9092, Volume 3, Issue 1, 2009.

[14] A. Ganesh, G. Balasubramanian “Simplified fuzzy adaptive resonance theory (SFAM) for tuberculosis response prediction” Excel India Publishers, New Delhi, India ISBN No. 93-80043-05-8 Page No.531-536.

[15] A. Ganesh, G. Balasubramanian “Dual Tree Complex Wavelet based Regularized Deconvolution for Medical Images” is accepted for publication by the Editor for journal Global Journal of Mathematical Sciences: Theory and Practical (GJMS).

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