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COMMON FIXED POINT THEOREMS IN MENGER PROBABILISTIC QUASI METRIC SPACES UNDER WEAK COMPATIBILITY

K.P.R. Sastry[1], Ch. Srinivasa Rao[2], K.K.M. sarma[3] and K. Satya Murty[4*]

[1]8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam -530 017, India

[2]Department of Mathematics, Mrs. A.V.N. College, Visakhapatnam -530 001, India
[3]Department of Mathematics, Andhra University, Visakhapatnam- 530003, India
[4]Department of Mathematics, Pydah College, Visakhapatnam - 530 002, India

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ABSTRACT

In this paper we prove a common fixed point theorem for weakly compatible maps in a Menger probabilistic quasi metric space. Incidentally, we observe that the result of Sunny Chauhan [12] is not valid through an example and modify it suitably.

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1. INTRODUCTION

Menger [7] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [11]. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points; but we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be of interest in the investigation of physical quantities and physiological threshold. It is also of fundamental importance in probabilistic functional analysis, non-linear analysis and applications [1, 2, 6].

In this section, some definitions and results in the theory of Menger probabilistic quasi metric spaces (briefly, Menger PQM-space) are given to fill in some background. For further information we refer to [3, 8, 10].

Definition 1.1 [11]: A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangle norm or t-norm if T satisfies the following conditions:

(1) $T(a,1) = a$ for all $a \in [0,1]$
(2) $T(a,b) = T(b,a)$
(3) $T(c,d) \geq T(a,b)$ for $c \geq a,\ d \geq b$
(4) $T(T(a,b),c) = T(a,T(b,c))$ for all $a,b,c \in [0,1]$

The following are the four basic t-norms:
$(i) T_M(x,y) = \text{Min}\{x,y\}$, $(ii)\ T_P(x,y) = x.y$,
$(iii) T(x,y) = \text{Max}\{x+y-1, 0\}$

Each t-norm T can be extended [10] (by associativity) in a unique way to an n-ary operation taking for $(x_1,..,x_n) \in [0,1]^n$ $(n \in N)$ the values

$$T^1(x_1,x_2) = T(x_1,x_2) \text{ and } T^n(x_1,\dots,x_{n+1}) = T(T^{n-1}(x_1,\dots,x_n),\ x_{n+1}) \text{ for } n \geq 2\ .$$

Definition 1.2: A t-norm T is of Hadzic-type ($\mathcal{H}$ -type,in short) and $T \in \mathcal{H}$ if the family $\{T^n\}_{n\in N}$ of its iterates defined, for each x in [0, 1], by $T^0(x) = 1,\ T^{n+1}(x) = T(T^n(x), x)$, for all $n \geq 0$, is equicontinuous at $x = 1$, that is given $\varepsilon \in (0,1)$, $\exists \delta \in (0,1)$ such that $x > 1-\delta \Longrightarrow T^n(x) > 1-\epsilon$, for all $n \geq 1$.

***Corresponding author:** *K. Satya Murty[4*]*
[4]Department of Mathematics, Pydah College, Visakhapatnam - 530 002, India

There is a nice characterization of continuous t-norm T of the class $\mathcal{H}$ [9].

Definition1.3 [4]: If T is a t-norm and $(x_1 x_2, \dots, x_n) \in [0,1]^n$ $(n \in N)$, then $T_{i=1}^{n} x_i$ is defined recurrently by 1, if $n = 0$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 1$. If $(x_i)_{i \in N}$ is a sequence of numbers from [0, 1], then $T_{i=1}^{\infty} x_i$ is defined as $\lim_{n\to\infty} T_{i=1}^{n} x_i$ (this limit always exists) and $T_{i=n}^{\infty} x_i$ as $T_{i=1}^{\infty} x_{n+i}$.

Proposition 1.4: Let $\{x_n\}_{n \in N}$ be a sequence of numbers from [0, 1] such that $\lim_{n\to\infty} x_n = 1$ and t-norm T be of $\mathcal{H}$-type. Then $\lim_{n\to\infty} T_{i=n}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1$.

Definition 1.5: A mapping F: $\mathbb{R}^+ \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

The Heaviside function H is a distribution function defined by $H(t) = \begin{cases} 0 \; if \; t \leq 0 \\ 1 \; if \; t > 0 \end{cases}$

The set of all distribution functions is denoted by $\mathfrak{D}$ and $\mathfrak{D}^+ = \{F \in \mathfrak{D} \,/\, F(0) = 0\}$.

Definition 1.6 [8,10]: A Menger PQM-space is a triple (X, F, T) where X is a non empty set, T is a continuous t-norm and F is a mapping from $X \times X$ into $\mathfrak{D}^+$ such that, if $F_{p,q}$ denotes the value of F at (p, q), then the following conditions hold:

(PQM1) $F_{p,q}(t) = F_{q,p}(t) = 1$ for all $t > 0$ iff $p = q$.
(PQM2) $F_{p,q}(t + s) \geq T(F_{p,r}(t), F_{r,q}(s)$ for all $p, q, r \in X$ and $t, s > 0$.

Definition 1.7 [8, 10]: Let (X, F, T) be a Menger PQM-space.

(i) A sequence $\{x_n\}$ is said to be F-convergent to $x \in X$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $k \in N$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq k$.
(ii) A sequence $\{x_n\}$ in X is called left Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $k \in N$ such that $F_{x_r,x_s}(\varepsilon) > 1 - \lambda$ for all $s \geq r \geq k$.
(iii) A Menger PQM-space (X, F, T) is called left complete if every left Cauchy sequence is F-convergent to a point in X.

In 1998, Jungck and Rhoades [5] introduced the following concept of weak compatibility.

Definition1.8 [5]: Let A and S be mappings from a Menger PQM-space (X, F, T) into itself. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$.

Sunny Chauhan [12] proved the following theorem.

Theorem 1.9: ([12], Theorem2.1) Let A, B, S, T and L be self maps of a complete Menger probabilistic quasi metric space (X, F, T) and suppose the following conditions are satisfied;

a) $AB(X) \cup ST(X) \subseteq L(X)$
b) $L(X)$ is a complete subspace of X
c) The pairs (L, AB) and (L, ST) are weakly compatible
d) $min\{F_{ABx,STy}(t), F_{STx,ABy}(t)\} \geq 1 - \alpha(t)(1 - F_{Lx,Ly}(t))$

for all $x, y \in X$ and every $t > 0$, where $\alpha: R \to (0,1)$ is a monotonic increasing function.

If $\lim_{n\to\infty} T_{i=n}^{\infty} \left(1 - \alpha^i(t)\right) = 1$, then A, B, S, T and L have a unique common fixed point.

The above theorem is not valid.

This is shown in the following example:

Example 1.10: If X = [0, 1] and $Ax = 0, Bx = 1$ and $Lx = x$ for all x.

Also take $S = A$ and $T = B$. Then $ABx = A1 = 0 \;\; \forall x$

$BA\,x = B0 = 1 \; \forall x$. $F_{ABx,ABy}(t) = F_{0,0}(t)$ =1and (AB, L) is weakly compatible.

Hence conditions (a), (b), (c) and (d) of Theorem 1.9 are satisfied for any relevant α and F. But A, B, S, T and L do not have a common fixed point.

2. MAIN RESULTS

We first prove a lemma.

Lemma 2.1: Suppose T is a continuous t- norm, $\{x_n\}$ is a sequence in [0, 1] and $\lim_{n\to\infty} T_{i=n}^{\infty} x_i = 1$. Then $x_n \to 1$.

Proof: Let $\varepsilon > 0$ .Then by hypothesis, there exists a positive integer N such that

$$T_{i=n}^{\infty} x_i > 1-\varepsilon \text{ for all } n \geq N$$

so that $T_{i=n}^{n+k} x_i > 1-\varepsilon$ for all n ≥ N and for every k ≥ 0

Consequently, $x_n = T_{i=n}^{n} x_i > 1-\varepsilon$ for every n ≥ N.

Hence $x_n \to 1$ as n → ∞ .

Notation 2.2: Throughout the rest of the paper, T stands for a continuous t – norm which satisfies the condition:

$\lim_{n\to\infty} T_{i=n}^{\infty} (1-\alpha^i(t)) = 1$ whenever α : (0, ∞) → (0, 1) …………(I)

From the above lemma, it follows that $\alpha^n(t) \to 1$ as n → ∞ if T satisfies the above condition (I).

Lemma 2.3: Let (X, F,T) be a Menger PQM-space. If a sequence $\{x_n\}$ in X is such that for every n ∈ N

$$F_{x_n,x_{n+1}}(t) \geq 1-\alpha^n(t)(1-F_{x_0,x_1}(t)) \text{ for every } t > 0,$$

where α: (0,∞) → (0,1) is a monotonic increasing function then the sequence $\{x_n\}$ is a left Cauchy sequence.

Proof: For every $m > n$, write m = n + k. Let ε > 0 and t > 0 .Then

$$\begin{aligned} F_{x_n,x_{n+k}}(t) &\geq T_{i=0}^{k-1}\left(F_{x_{n+i},x_{n+i+1}}\left(\tfrac{t}{k}\right)\right) \\ &\geq T_{i=0}^{k-1}\left(1-\alpha^{n+i}\left(\tfrac{t}{k}\right)\left(1-F_{x_0,x_1}\left(\tfrac{t}{k}\right)\right)\right) \\ &\geq T_{i=0}^{k-1}\left(1-\alpha^{n+i}\left(\tfrac{t}{k}\right)\right) \\ &\geq T_{j=n}^{n+k-1}\left(1-\alpha^{j}\left(\tfrac{t}{k}\right)\right) \quad \text{(put n + i = j)} \\ &= T_{j=n}^{m-1}\left(1-\alpha^{j}\left(\tfrac{t}{m-n}\right)\right) \\ &\geq T_{j=n}^{m-1}\left(1-\alpha^{j}(t)\right) \quad (\because \alpha \text{ is monotonic increasing}) \\ &\geq T_{j=n}^{\infty}\left(1-\alpha^{j}(t)\right) \\ &> 1-\varepsilon \quad \text{(for } n \geq N\text{, since (I) holds)} \end{aligned}$$

Hence the sequence $\{x_n\}$ is a left Cauchy sequence.

Corollary 2.4: Let (X, F, T) be a Menger PQM-space. If the sequence $\{x_n\}$ in X is such that for every n ∈ N

$$F_{x_n,x_{n+1}}(t) \geq 1-\alpha^n(t) \quad \text{for every } t > 0,$$

where α : (0,∞) → (0,1) is a monotonic increasing function, then the sequence $\{x_n\}$ is a left Cauchy sequence.

Proposition 2.5: If (X,F,T) is a PQM-space then (X,E,T) is a Menger probabilistic metric space where $E_{x,y}(t) = \min\{F_{x,y}(t), F_{y,x}(t)\}$

Proof: $E_{x,y}(t) = \min\{F_{x,y}(t), F_{y,x}(t)\}$ for $x, y \in X$

Clearly $E_{x,y}(t) = E_{y,x}(t)$

$$\begin{aligned} E_{x,y}(t) = 1 &\Leftrightarrow F_{x,y}(t) = 1 \text{ and } F_{y,x}(t) = 1 \\ &\Leftrightarrow x = y \end{aligned}$$

$$\min\{F_{x,z}(t+s), F_{z,x}(t+s)\} \geq T\left(E_{x,y}(t), E_{y,z}(s)\right)$$
$$= T\left(min\{F_{x,y}(t), F_{y,x}(t)\}, min\{F_{y,z}(s), F_{z,y}(s)\}\right)$$

$and\ F_{x,z}(t+s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right) \geq T\left(E_{x,y}(t), E_{y,z}(s)\right)$

$F_{z,x}(s+t) \geq T\left(F_{z,y}(t), F_{y,x}(t)\right) \geq T\left(E_{z,y}(s), E_{y,x}(t)\right)$

$\therefore E_{x,z}(t+s) \geq T\left(E_{x,y}(t), E_{y,z}(s)\right)$

$\therefore$ If (X, F, T) is a PQM-space then (X, E, T) is a Menger probabilistic metric space.

Definition 2.6: E is called the induced probabilistic metric on X induced by the quasi probabilistic metric F.

Theorem 2.7: Let (X, F, T) be a complete Menger probabilistic metric space and let $g, L: X \to X$ be maps that satisfy the following conditions:

(a) $g(X) \subseteq L(X)$;
(b) $L(X)$ is a complete subspace of X;
(c) $F_{gx,gy}(t) \geq 1 - \alpha(t)\left(1 - F_{Lx,Ly}(t)\right)$ for each $t > 0$ and for all $x, y \in X$, where $\alpha: (0, \infty) \to (0,1)$ is a monotonic increasing function.
If $\lim_{n\to\infty} T_{i=n}^{\infty} (1 - \alpha^i(t)) = 1$ then g and L have a unique point of coincidence. If further
(d) The pair (L,g) is weakly compatible, then L and g have a unique common fixed point.

Proof: $Let\ x_0 \in X$. From condition (a), we can find x_1, such that $L(x_1) = g(x_0)$.

Inductively, we define a sequence $\{x_n\}$ such that $Lx_{n+1} = gx_n$ for n = 0, 1, 2, …

Now, by taking $x = x_{n-1}$ and $y = x_n$ in (c), we get, for t > 0,

$$\begin{aligned} F_{gx_{n-1},gx_n}(t) &\geq 1 - \alpha(t)\left(1 - F_{Lx_{n-1},Lx_n}(t)\right) \\ &\geq 1 - \alpha(t)\left(1 - \left(1 - \alpha(t)\left(1 - F_{Lx_{n-2},\,Lx_{n-1}}(t)\right)\right)\right) \\ &\geq 1 - \alpha^2(t)\left(1 - F_{Lx_{n-2},Lx_{n-1}}(t)\right) \end{aligned}$$

and by induction we get

$$F_{Lx_n,Lx_{n+1}}(t) \geq 1 - \alpha^n(t)\left(1 - F_{Lx_0,Lx_1}\right) \text{ for } n = 1,2, ...$$

Now by Lemma 2.3, $\{Lx_n\}$ is a left Cauchy sequence.

Since the space $L(X)$ is left complete,

there exists $z \in L(X)$, such that $\lim_{n\to\infty} Lx_n = z$.

Hence $\lim_{n\to\infty} gx_{n-1} = \lim_{n\to\infty} Lx_n = z$.

Since $z \in L(X)$, it follows that there exists $v \in X$ such that $L(v) = z$.

We prove that $gv = z$.

Put $x = x_{n-1}$ and $y = v$ in(c). We get

$F_{gx_{n-1},gv}(t) \geq 1 - \alpha(t)\left(1 - F_{Lx_n,Lv}(t)\right)$

On letting $n \to \infty$ we get

$F_{z,gv}(t) \geq 1 - \alpha(t)\left(1 - F_{z,z}(t)\right)$

so that $F_{z,gv}(t) = 1$. This is true for all t > 0.

$$\therefore gv = z.$$

Thus v is a point of coincidence to g and L. Suppose $gx = Lx$.

Then, $F_{gx,gv}(t) \geq 1 - \alpha(t)\big(1 - F_{Lx,Lv}(t)\big)$
$= 1 - \alpha(t)\big(1 - F_{gx,gv}(t)\big)$

So that $0 \geq (1 - \alpha(t))\,\big(1 - F_{gx,gv}(t)\big)$

Consequently $1 - F_{gx,gv}(t) = 0 \quad \forall\, t > 0$.

Similarly, we can show that $1 - F_{gv,gx}(t) = 0 \quad \forall\, t > 0.$

Hence $gx = gv$. Thus g and L have a unique coincidence point.

Suppose the pair (L, g) is weakly compatible.

Then $gv = z = Lv$ implies that $Lg(v) = gL(v)$ so that $Lz = gz$.

Thus, z and gz are coincidence points of g and L.

Hence $gz = z$ so that $Lz = gz = z$.

Consequently z is a common fixed point of L and g.

Since, under (d) every fixed point is a coincidence point, L and g have unique common fixed point.

Now we have the following corollary which can be treated as a modification of Theorem 1.9.

Corollary 2.8: Let A, B, S, T and L be self maps of a complete Menger probabilistic quasi metric space(X, F, T) and suppose the following conditions are satisfied:

(i) $AB(X) \cup ST(X) \subseteq L(X)$;
(ii) $L(X)$ is a complete subspace of$X$;
(iii) The pairs (L, AB) and (L, ST) are weakly compatible;
(iv) $min\{F_{ABx,STy}(t), F_{STx,ABy}(t)\} \geq 1 - \alpha(t)\big(1 - F_{Lx,Ly}(t)\big)$for all $x, y \in X$ and every $t > 0$, where $\alpha: (0, \infty) \to (0,1)$ is a monotonic increasing function.
(v) A, B and L commute and S, T and L commute.

If $\lim_{n\to\infty} T_{i=n}^{\infty}\,(1 - \alpha^i(\mathrm{t})) = 1$, then A, B, S, T and L have a unique common fixed point.

Proof: If we put $x = y$ in condition (iv) of the above theorem we obtain that

$$min\{F_{ABy,STy}(t), F_{STy,ABy}(t)\} \geq 1 - \alpha(t)\big(1 - F_{Ly,Ly}(t)\big)$$

so that $F_{ABy,STy}(t) = 1$ and $F_{STy,ABy}(t) = 1$

Hence $ABy = STy$ for all $y \in X$.

Thus $AB = ST$.

If we take, $AB = g$ in Theorem 2.7, AB and g have a unique common fixed point, say z.

Then AB z = L z = z

So that $AB(A\,z) = AABz = ALz = Az = ALz = L(Az)$.

Thus Az is also fixed point of AB and L.

Similarly Bz is also fixed point of AB and L.

By the uniqueness of fixed point, follows that $Az = z = Bz = Lz$.

Thus z is the unique common fixed point of A, B and L.

Similarly we can show that z is the unique common fixed point of S, T and L. (since AB = ST).

Thus A, B, S, T and L have unique common fixed point.

NOTE:

(i) It may be observed that in Example 1.10, A and B do not commute.

(ii) Even through AB = ST in corollary 2.8, it can be shown that A, B, S, T may be different, in view of the following Example.

Example 2.9: Let X = [0, 1], $Ax = 0, Bx = x^3, Sx = 0, Tx = x^2$ $and\ Lx = x$ for all $x \in X$. Then $AB = ST$ on X and all the hypotheses of corollary 2.8 is satisfied. But $B \neq T$ so that the triad (A, B, L) is not equal to the triad (S, T, L).

However A, B, S, T and L have unique common fixed point, namely, 0.

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