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EXPLICIT PROJECTION METHODS FOR VARIATIONAL INEQUALITIES

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ABSTRACT

We present two explicit methods for solving variational inequalities. Each iteration of the proposed methods consists of projection onto a halfplace containing the feasible set. Our projection is easy to calculate. Moreover, we find the step size through an Armijo-like search instead of defining them exogenously. Our methods are proved to be globally convergent under pseudomontonicity and continuity of the operator. No coerciveness, paramonotonicity or Lipschitz continuity assumption is imposed, thus we generalize some recent results in the literature.

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Key words: *Variational inequalities, Pseudomontone operator, Explicit projection algorithm, Global convergence.*

1. INTRODUCTION

Let $f : R^n \to R^n$ be a continuous mapping and $C \subset R^n$ be a nonempty closed convex set. The inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. We consider the following variational inequality problem, denoted by VI(f, C), is to find a vector $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \qquad \forall\, y \in C. \tag{1}$$

Let S denote the solution set of problem (1). Throughout this paper, we assume that S is nonempty.

The variational inequality problem was first introduced by Hartman and Stampacchia [1] in 1966. In recent years, many iterative projection-type algorithms have been proposed and analyzed for solving the variational inequality problem, see [2], and the references therein. To implement these algorithms one has to find the projection onto the feasible set C, which is not possible except in some simple cases(e.g., C is a halfspace or a ball). If so, the overall efficiency of a projection method will be seriously affected.

To overcome this difficulty, some inexact projection algorithms for solving VI(f, C) were proposed, see [3-5]. Among them, the relaxed projection algorithm for solving VI(f, C) proposed by Fukushima [3] is quite attractive. Specifically, each iteration of the proposed algorithm consists of projection onto a halfspace containing the feasible set rather than the latter set itself. Observe that projection onto halfspace is easily computable, which can be explicitly

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represented without resorting to projection operator. However, the conditions for the convergence of the method, i.e., strong monotonicity and coerciveness assumptions, are stringent, which precludes the application of the method in reality. To extend the Fukushima's algorithm to a scope. Recently, Yang [6] proposed a relaxed projection algorithm for solving VI(f,C) and established the global convergence under weaker conditions. Mainly the strong monotonicity of f is replaced by the weak co-coercivity. Most recently, Censor, Gibali and Reich [7] presented two extensions of Korpelevich's extragradient method for solving VI(f,C) and established the global convergence under f being Pseudomontone and Lipschitz continuous. Bello Cruz and Iusem [8,9] extended the Fukushima's algorithm to solving generalized variational inequality problems, under maximal monotone, paramonotone and other assumptions, the global convergence of algorithm is proved. Note that the step size was exogenously defined in the literature [3, 6, 8, 9]. That is to say, their step size isn't associated with the current iterate.

In this paper, we would introduce two new explicit projection algorithms inspired by the work of literature [10] for solving the split feasibility problem. Our methods possess the following properties: (a) Our projection is onto halfspace and can be explicitly represented; (b) the proof of the global convergence need only f to be pseudomontone and continuous; (c) Our step size is defined through an Armijo-like search instead of defining them exogenously.

2. PRELIMINARIES

In this section, we recall some useful definitions and results which will be used in this paper.

For a nonempty closed convex set $\Omega \subset R^n$ and a vector $x \in R^n$, the projection of x onto Ω is defined as:

$$P_\Omega(x) = \arg\min \{\|y-x\| \mid y \in \Omega\}.$$

We have the following properties on the projection operator, see [11].

Lemma 2.1: Let $\Omega \subset R^n$ be a closed convex set. Then for any $x \in R^n$ and $z \in \Omega$,

(1) $\|P_\Omega(x)-z\|^2 \le \|x-z\|^2 - \|P_\Omega(x)-x\|^2$;

(2) $\langle P_\Omega(x)-x,\, z-P_\Omega(x)\rangle \ge 0.$

Let $f : R^n \to R^n$ be a mapping. For any $x \in R^n$ and $\alpha > 0$, define

$$e(x,\alpha) = x - P_\Omega(x-\alpha f(x)).$$

Lemma 2.2: Let $f : R^n \to R^n$ be a continuous mapping. For any $x \in R^n$ and $\alpha > 0$, we have

$$\min\{1,\alpha\}\|e(x,1)\| \le \|e(x,\alpha)\| \le \max\{1,\alpha\}\|e(x,1)\|$$

Remark 2.1: From the nondecreasing property of $\|e(x,\alpha)\|$ on $\alpha > 0$ by Toint[12] and the nonincreasing property of $\|e(x,\alpha)\|/\alpha$ on $\alpha > 0$ by Gafini and Bertsekas [13], Lemma 2.2 is easily proved.

Lemma 2.3: If $f : R^n \to R^n$ is a continuous mapping and $\|e(x,1)\| \ne 0$, then there exists $0 < L < 1$ and $\bar{\alpha} > 0$ such that for all $0 < \alpha \le \bar{\alpha}$, it holds that

$$\alpha\| f(x)-f(x-e(x,\alpha))\| \le L\|e(x,\alpha)\|.$$

Proof: See Lemma 3.1 in [14].

Definition 2.1: A mapping $f : R^n \to R^n$ is said to be pseumonotone if

$$\langle f(y), x-y\rangle \ge 0 \quad \Rightarrow \quad \langle f(y), x-y\rangle \ge 0 \quad \forall x, y \in R^n.$$

Remark 2.2: we know the pseumonotonicity is weaker than paramonotonicity (see [8, 9]) or Monotonicity.

3. TWO EXPLICIT PROJECTION ALGORITHMS

In this paper, we assume that the convex set C satisfies the following assumptions:
(1) The set C is given by

$$C = \{x \in R^n | c(x) \le 0\}. \tag{2}$$

Where $c : R^n \to R$ is convex (not necessarily differentiable) function and C is nonempty.

(2) For any $x \in R^n$, at least one subgradient $\xi \in \partial c(x)$ can be calculated, where $\partial c(x)$ is the subdifferential of $c(x)$ at x and is defined as follows:

$$\partial c(x) = \{\xi \in R^n | c(z) \ge c(x) + \langle \xi, z-x\rangle, \forall z \in R^n\}.$$

Note that the differentiability of $c(x)$ is not assumed, therefore the set C is quite general.

For example, any system of inequalities $c_j(x) \le 0$, $j \in J$, where $c_j(x)$ is convex and J is an arbitrary index set, is the same as the single inequality $c(x) \le 0$ with $c(x) = \sup\{ c_j(x) \mid j \in J \}$.

The following lemma provides an important boundedness property of the subdifferential.

Lemma 3.1: If $c : R^n \to R$ is a convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded on any bounded subset of R^n.

Proof: See [15].
Denote

$$C_k = \{x \in R^n | c(x^k) + \langle \xi^k, x-x^k\rangle \le 0\}, \tag{3}$$

where $\xi^k \in \partial c(x^k)$.

Proposition 3.1: For every nonnegative integer k, let $x^k \in R^n$ and C_k be defined as in (3). Then for any $x \in R^n$, we have

$$P_{C_k}(x) = \begin{cases} x - \dfrac{c(x^k) + \langle \xi^k, x-x^k\rangle}{\|\xi^k\|^2}\xi^k, & if\ c(x^k) + \langle \xi^k, x-x^k\rangle > 0; \\ x, & otherwise. \end{cases} \tag{4}$$

Proof: See [16].

Remark 3.1: (1) From the definition of subdifferential, we have $C \subseteq C_k$ for all k. In fact, for any $x \in C$ and $\xi^k \in \partial c(x^k)$, we have

$$c(x^k) + \langle \xi^k, x - x^k \rangle \le c(x) \le 0,$$

i.e., $x \in C_k$ and hence $C \subseteq C_k$.

(2) From proposition 3.1, we can observe that P_{C_k} can be explicitly represented without resorting to projection operator, thus its computation is easy. Recently, C_k is often regared as the projection region in the algorithm of the split feasibility problem, see [17-20].

Algorithm 3.1: Choose an intial point x^0, parameters β, l $\in$ (0, 1), γ > 0 and set k = 0.

Step 1: Choose $\xi^k \in \partial c(x^k)$. Let

$$C_k = \left\{ x \in R^n \middle| c(x^k) + \langle \xi^k, x - x^k \rangle \le 0 \right\}$$

and

$$\overline{x}^k = P_{C_k}[x^k - \alpha_k f(x^k)].$$

Where $\alpha_k = \gamma l^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$\alpha_k \left\| f(x^k) - f(\overline{x}^k) \right\| \le \beta \left\| x^k - \overline{x}^k \right\|. \tag{5}$$

Stop if $x^k = \overline{x}^k$; otherwise, go to Step 2.

Step 2: Let $\{\varepsilon_k\}$ be a sequence which satisfies

$$\varepsilon_k > 0, \quad \sum_{k=0}^{\infty} \varepsilon_k^2 < +\infty.$$

Find x^{k+1} satisfying

$$\left\| x^{k+1} - P_{C_k}[x^k - \alpha_k f(\overline{x}^k)] \right\| \le \varepsilon_k \left\| x^k - \overline{x}^k \right\|. \tag{6}$$

Let $k := k + 1$ and go to Step 1.

Remark 3.2: (1) By Proposition 3.1, the projection onto C_k can be directly calculated, so this algorithm can be easily implemented.

(2) In our algorithm, the step size α_k is obtained through an Armijo-like search, which is associated with the current iterative point x^k.

(3) By Lemma 2.3, our Armijo-like search procedure (5) is well defined. Moreover, inf{ α_k } also exists and inf{ α_k } > 0. On the other hand, from the definition of α_k, we have $\alpha_k \le \gamma$, for all k.

(4) If $x^k = \bar{x}^k$ for some positive integer k, then x^k is a solution of problem (1). In fact, suppose $x^k = \bar{x}^k$. Since $\bar{x}^k \in C_k$, it follows that

$$c(x^k) + \left\langle \xi^k, \bar{x}^k - x^k \right\rangle \le 0,$$

where $\xi^k \in \partial c(x^k)$, which implies that $c(x^k) \le 0$, i.e., $x^k \in C$. By the definition of $\bar{x}^k$ and Lemma 2.1 (2), we obtain

$$\left\langle x - x^k, x^k - \alpha_k f(x^k) - \bar{x}^k \right\rangle \le 0, \forall x \in C_k,$$

which, together with $C \subseteq C_k$ and $x^k = \bar{x}^k$, implies that

$$\left\langle x - x^k, f(x^k) \right\rangle \ge 0, \forall x \in C,$$

this means that x^k is a solution of problem (1).

Algorithm 3.1 is called inexact projection one. Setting $x^{k+1} = P_{C_k}[x^k - \alpha_k f(\bar{x}^k)]$ in Algorithm 3.1, we obtain the following explicit extragradient-type projection method:

Algorithm 3.2: Choose an intial point x_0, parameters β, l $\in$ (0, 1), γ > 0 and set $k = 0$.

Step 1: Choose $\xi^k \in \partial c(x^k)$. Let

$$C_k = \left\{ x \in R^n \middle| c(x^k) + \left\langle \xi^k, x - x^k \right\rangle \le 0 \right\}$$

and

$$\bar{x}^k = P_{C_k}[x^k - \alpha_k f(x^k)].$$

Where $\alpha_k = \gamma l^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$\alpha_k \left\| f(x^k) - f(\bar{x}^k) \right\| \le \beta \left\| x^k - \bar{x}^k \right\|.$$

Stop if $x^k = \bar{x}^k$; otherwise, go to Step 2.

Step 2: Set

$$x^{k+1} = P_{C_k}[x^k - \alpha_k f(\bar{x}^k)].$$

Stop if $x^k = x^{k+1}$; otherwise, let $k = k+1$, go to Step 1.

Remark 3.3: (1) If $x^k = x^{k+1}$ for some positive integer k, then x^k is a solution of problem(1)

(1) whose proof is similar to Remark 3.2 (4).

(2) Algorithm 3.2 can be regarded as a modification of the extragradient method proposed by Korpelevich [21], the main modification is to use the halfspace C_k in place of the closed convex set C. Due to the special from of C_k, the projection onto C_k for each k can be directly calculated, thus Algorithm 3.2 can be easily implemented. Therefore, Algorithm 3.2 improves the one proposed by Korpelevich [21].

4. CONVERGENCE

Now, we turn to consider the convergence of Algorithm 3.1 and Algorithm 3.2. Certainly, if algorithm terminates within finite steps, e.g., step k, then x^k is a solution of problem (1). So, in the following analysis, we assume that our algorithm always generates an infinite sequence.

Theorem 4.1: Let {x^k} be a sequence generalized by Algorithm 3.1. If f is continuous, pseudomontone and the solution set of VI (f, C) is nonempty, then {x^k} converges to a solution of VI (f, C).

Proof. Let $\sigma = \beta^2 - 1$. Then σ < 0. Since $\sum_{k=0}^{\infty} \varepsilon_k^2 < +\infty$, we have

$$\prod_{k=0}^{\infty}(1+(-\frac{2}{\sigma}\varepsilon_k^2)) < +\infty .$$

Denote $D_d = \prod_{k=0}^{\infty}(1+\delta_k^2)$, where $\delta_k = \sqrt{-\frac{2}{\sigma}}\varepsilon_k$. Let x^* be a solution of problem (1). Then we have

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 = &\left\|x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]\right\|^2 + \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 \\ &+2\left\langle x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)], P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\rangle .\end{aligned} \tag{7}$$

Since

$$\begin{aligned}&2\left\langle x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)], P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\rangle \\ &\le \frac{1}{\delta_k^2}\left\|x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]\right\|^2 + \delta_k^2\left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 .\end{aligned}$$

This, together with (7), yields that

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 \le &(1+\delta_k^2)\left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 \\ &+(1+\frac{1}{\delta_k^2})\left\|x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]\right\|^2\end{aligned} \tag{8}$$

Since

$$\left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 \le \left\|x^k-\alpha_k f(\bar{x}^k)-x^*\right\|^2 - \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k+\alpha_k f(\bar{x}^k)\right\|^2$$

$$= \left\|x^k-x^*\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), x^k-x^*\right\rangle - \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\|^2$$

$$-2\alpha_k \left\langle f(\bar{x}^k), P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\rangle$$

$$= \left\|x^k-x^*\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), x^k-\bar{x}^k\right\rangle - 2\alpha_k \left\langle f(\bar{x}^k), \bar{x}^k-x^*\right\rangle$$

$$- \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\rangle$$

$$\le \left\|x^k-x^*\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), x^k-\bar{x}^k\right\rangle - \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\|^2$$

$$-2\alpha_k \left\langle f(\bar{x}^k), P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^k\right\rangle$$

$$= \left\|x^k-x^*\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\rangle$$

$$- \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k+\bar{x}^k-x^k\right\|^2$$

$$= \left\|x^k-x^*\right\|^2 - 2\alpha_k \left\langle f(\bar{x}^k), P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\rangle -$$

$$- \left\|x^k-\bar{x}^k\right\|^2 + 2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, x^k-\bar{x}^k\right\rangle$$

$$= \left\|x^k-x^*\right\|^2 - \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\|^2 - \left\|x^k-\bar{x}^k\right\|^2$$

$$+2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, x^k-\bar{x}^k-\alpha_k f(\bar{x}^k)\right\rangle \quad , \tag{9}$$

where the first inequality is from Lemma 2.1 (1) and the second one is from the pseudomonotonicity of f .

By taking $x = x^k - \alpha_k f(x^k)$ and $z =$ $P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]$ in Lemma 2.1 (2), we obtain

$$\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, \bar{x}^k-x^k+\alpha_k f(x^k)\right\rangle \ge 0. \tag{10}$$

Combining (10) and search procedure (5), we obtain

$$2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, x^k-\bar{x}^k-\alpha_k f(\bar{x}^k)\right\rangle$$

$$\le 2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, x^k-\bar{x}^k-\alpha_k f(\bar{x}^k)\right\rangle + 2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, \bar{x}^k-x^k+\alpha_k f(x^k)\right\rangle$$

$$= 2\alpha_k \left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, f(x^k)-f(\bar{x}^k)\right\rangle$$

$$\le 2\alpha_k \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\| \left\|f(x^k)-f(\bar{x}^k)\right\|$$

$$\le \alpha_k^2 \left\|f(x^k)-f(\bar{x}^k)\right\|^2 + \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\|^2$$

$$\le \beta^2 \left\|x^k-\bar{x}^k\right\|^2 + \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\|^2, \tag{11}$$

which, together with (6),(8) and (9), we conclude that

$$\left\|x^{k+1}-x^*\right\|^2 \le (1+\delta_k^2)\left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 +(1+\frac{1}{\delta_k^2})\left\|x^{k+1}-P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]\right\|^2$$

$$\le (1+\delta_k^2)\left\|x^k-x^*\right\|^2 +(1+\frac{1}{\delta_k^2})\,\varepsilon_k^2\left\|x^k-\bar{x}^k\right\|^2 +(1+\delta_k^2)(\beta^2-1)\left\|x^k-\bar{x}^k\right\|^2$$

$$= (1+\delta_k^2)\left\|x^k-x^*\right\|^2 +[(1+\frac{1}{\delta_k^2})\,\varepsilon_k^2+\sigma\,(1+\delta_k^2)]\left\|x^k-\bar{x}^k\right\|^2$$

$$= (1+\delta_k^2)\left\|x^k-x^*\right\|^2 +(\frac{\sigma}{2}-\varepsilon_k^2)\left\|x^k-\bar{x}^k\right\|^2. \qquad (12)$$

which can deduce that $\left\|x^{k+1}-x^*\right\|^2 \le D_d\left\|x^0-x^*\right\|^2$ because $\sigma < 0$. It follows from $D_d < +\infty$ that { x^k } is a bounded sequence. Consequently, we obtain from (12) and $\sum_{k=0}^{\infty}\varepsilon_k^2 < +\infty$ that $\left\{\left\|x^k-x^*\right\|\right\}$ is a convergent sequence and

$$\lim_{k\to\infty}\left\|x^k-\bar{x}^k\right\|^2 = 0. \qquad (13)$$

Assume that $\hat{x}$ is an accumulation point of { x^k } and $x^{k_i}\to\hat{x},\ i\to\infty$, where $\left\{x^{k_i}\right\}_{i=1}^{\infty}$ is a subsequence of { x^k }.

Now, we show that $\hat{x}$ is a solution of the problem (1).

First, we show that $\hat{x}\in C$. By $\bar{x}^{k_i} = P_{C_{k_i}}(x^{k_i}-\alpha_{k_i} f(x^{k_i}))\in C_{k_i}$ and the definition of C_{k_i} , we have

$$c(x^{k_i})+\left\langle \xi^{k_i},\bar{x}^{k_i}-x^{k_i}\right\rangle \le 0,\ \forall i=1,2,...,$$

where $\xi^{k_i}\in\partial c(x^{k_i})$. Passing to the limit $k_i\to\infty$ in the above inequality, we deduce, by (13) and Lemma 3.1, that

$c(\hat{x}) \le 0$, that is, $\hat{x}\in C$.

By the Lemma 2.3, we obtain that the parameter sequence { α_k } is bounded below from zero, i.e., inf{ α_k } > 0. It follows from Lemma 2.2 and (13) that

$$\lim_{k_i\to\infty}\left\|e(x^{k_i},1)\right\| \le \lim_{k_i\to\infty}\frac{\left\|x^{k_i}-\bar{x}^{k_i}\right\|}{\min\left\{1,\alpha_{k_i}\right\}} \le \lim_{k_i\to\infty}\frac{\left\|x^{k_i}-\bar{x}^{k_i}\right\|}{\min\left\{1,\inf\left\{\alpha_k\right\}\right\}} = 0 \qquad .$$

(14)

By $x^{k_i}-e(x^{k_i},1) = P_{C_{k_i}}(x^{k_i}-f(x^{k_i}))$ and Lemma 2.1 (2), for all $x\in C_{k_i}$, we have

$$\left\langle x-x^{k_i}+e(x^{k_i},1),\ f(x^{k_i})-e(x^{k_i},1)\right\rangle \ge 0,$$

letting $k_i \to \infty$, taking into account (14), the continuity of f and $C \subseteq C_k$, we obtain

$$\left\langle f(\hat{x}), x-\hat{x} \right\rangle \geq 0, \quad \forall x \in C.$$

That is, $\hat{x}$ is a solution of problem (1).

Thus, we may use $\hat{x}$ in place of x^*. From (12), we know that $\left\{\left\|x^k-\hat{x}\right\|\right\}$ is a convergent sequence. Because there is a subsequence $\left\{\left\|x^{k_i}-\hat{x}\right\|\right\}$ of $\left\{\left\|x^k-\hat{x}\right\|\right\}$ converging to 0, then $x^k \to \hat{x}$ as $k \to \infty$. The proof is completed.

Remark 4.1: It is obvious that our conditions for the global convergence, i.e., pseudomontone and continuous, are weaker than ones of [3, 6-8].

Theorem 4.2: Let $\{x^k\}$ be a sequence generalized by Algorithm 3.2. If f is continuous, pseudomontone and the solution set of VI(f, C) is nonempty, then $\{x^k\}$ converges to a solution of VI(f, C).

Proof: By the definition of x^{k+1}, inequalities (9) and (11), we obtain

$$\begin{aligned}
\left\|x^{k+1}-x^*\right\|^2 &= \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-x^*\right\|^2 \\
&\leq \left\|x^k-x^*\right\|^2 - \left\|P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k\right\|^2 - \left\|x^k-\bar{x}^k\right\|^2 \\
&\qquad +2\left\langle P_{C_k}[x^k-\alpha_k f(\bar{x}^k)]-\bar{x}^k, x^k-\bar{x}^k-\alpha_k f(\bar{x}^k)\right\rangle \\
&\leq \left\|x^k-x^*\right\|^2 - (1-\beta^2)\left\|x^k-\bar{x}^k\right\|^2
\end{aligned}$$

since $\beta \in$ (0, 1), it follows that the sequence $\{\left\|x^k-x^*\right\|\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x^k\}$ is bounded and

$$\lim_{k\to\infty}\left\|x^k-\bar{x}^k\right\|^2 = 0.$$

Applying the similar proof of Theorem 3.1, we obtain the desired result.

Remark 4.2: Our conditions for the global convergence are weaker than those in [7] and [21], mainly the Lipschitz continuity of f is replaced by the continuity.

5. CONCLUSION

In this paper, two new explicit projection algorithms for the variational inequality problem have been presented. The main advantages of the proposed methods are that each iteration consists of the projection onto a halfspace implemented very easily, and the conditions for global convergence are weaker than those of the existing projection method for solving the problem.

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