ON DIGITAL HOMOTOPY OF DIGITAL PATHS

Ali Mutlu\(^a\), Berrin Mutlu\(^b\) and Simge Öztunç\(^a\)*

\(^a\)Celal Bayar University, Faculty of Science and Arts, Department of Mathematics
Muradiye Campus, 45047, Manisa, TURKEY

\(^b\)Hasan Türek Anatolian High School, Mathematics Teacher 45020, Manisa, TURKEY

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ABSTRACT

In this paper, we study digital homotopy of digital paths due to Laurence Boxer. We give some theorems, propositions and definitions on digital paths, digital path connectedness and introduce digital convex set and digital contractible spaces.

Keywords: Digital image, digital continuous maps, digital homotopy, digital path.


1. INTRODUCTION

In 1937, Alexandroff published a paper where the term ‘Discrete Topology’ was explicitly used in the title. E. Khalimsky investigated ordered connected topological spaces and wrote a book on this topic in 1977. Later on in collaboration of the New York Scholl of Topology (Kopperman, Meyer, Kong and others) it was realized that ordered connected spaces are very suitable for treating problems of digital topology. At the end of the eighties V. Kovalevsky gave a sound fundament for digital topology which concern with the part of Alexandroff theory. Similar theory was provided in the same time by G. Herman in 1979. A. Rosenfeld published a paper which had the title ‘Digital Topology’. Rosenfeld’s paper was very influential since for the first time some very difficult problems of digital topology were treated rigorously. His paper was based on results of Duda, Hart and Munson.

The notion of digital image, digital continuous map and digital homotopy studied in \([2, 3, 5, 6, 7 \text{ and } 9]\). Their recognition and efficient computation became a useful material for our study. In this paper we give the digital versions of some properties of the homotopy of path connected spaces in order to construct a tool for digital homotopy researchers. In section two we recall some properties and definitions from Boxer \([3]\) in order to use in section three. In section three we give digital homotopy properties of digital paths, defined convexity in digital images and then stated some results by using convexity in digital images.

2. PRELIMINARIES

In this section we recall some basic definitions, theorems and propositions from Boxer \([3]\). In this paper, we denote the set of integers by \(\mathbb{Z}\). Then \(\mathbb{Z}^n\) represents the set of lattice points in Euclidean \(n\) – dimensional spaces. A finite subset of \(\mathbb{Z}^n\) is called to be digital image.

A variety of adjacency relations are used in the digital image research. The following \([6]\) are commonly used. Two points \(p\) and \(q\) in \(\mathbb{Z}^2\) are 8 – adjacent if they are distinct and differ by at most 1 in each coordinate; \(p\) and \(q\) in \(\mathbb{Z}^2\) are 4 – adjacent if they are 8 – adjacent and differ in exactly one coordinate. Two points \(p\) and \(q\) in \(\mathbb{Z}^3\) are 26 – adjacent if they are distinct and differ by at most 1 in each coordinate; they are 18 – adjacent if they are 26 – adjacent and differ in at most two coordinates; they are 6 – adjacent if they are 18 – adjacent and differ in exactly one coordinate. For \(k \in \{4,8,6,18,26\}\), a \(k – \text{neighbor}\) of a lattice point \(p\) is a point that is \(k\) – adjacent to \(p\).

*Corresponding author: Ali Mutlu\(^a\)*

\(^a\)Celal Bayar University, Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa, TURKEY
4-adjacency in $\mathbb{Z}^2$ and 6-adjacency in $\mathbb{Z}^3$ are generalized by taking $p, q \in \mathbb{Z}^n$ are 2n-adjacent if $p \neq q$ and $p$ and $q$ differ by 1 in one coordinate and by 0 in all other coordinates.

More extensive adjacency relations are investigated in [4]. In the following, if $\kappa$ is an adjacency relation defined for an integer $\kappa$ on $\mathbb{Z}^n$ as one of the $k$-adjacencies discussed above, that is, if

$$(n,k) \in \{(1,2),(2,4),(2,8),(3,6),(3,18),(3,26)\} \text{ or } k = 2n$$

We assume $\kappa$-adjacency as $k$-adjacency, $\kappa$-connectedness as $k$-connectedness, etc.

Suppose that $\kappa$ be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa$-connected [4] if and only if for every pair of points $\{x, y\} \subset X$, $x \neq y$, there is a set $\{x_0, x_1, \ldots, x_r\} \subset X$ such that $x = x_0$, $x_r = y$ and $x_i$ and $x_{i+1}$ are $\kappa$-neighbors, $i \in \{0,1,\ldots,c-1\}$.

**Definition 2.1:** ([2]; see also [8]) Let $X$ and $Y$ are digital images such that $X \subset \mathbb{Z}^n$, $Y \subset \mathbb{Z}^n$. Assume that $f : X \rightarrow Y$ be a function. Let $\kappa_i$ be an adjacency relation defined on $\mathbb{Z}^n$, $i \in \{0,1\}$. $f$ is called to be $(\kappa_0, \kappa_1)$-continuous if the image under $f$ of every $\kappa_0$-connected subset of $X$ is $\kappa_1$-connected.

A function satisfying Definition 2.1 is referred to be digitally continuous. A consequence of this definition is given below.

**Definition 2.2:** ([2]; see also [8]) Let $X$ and $Y$ are digital images. Then the function $f : X \rightarrow Y$ is said to be $(\kappa_0, \kappa_1)$-continuous if and only if for every $\{x, y\} \subset X$ such that $x_0$ and $x_1$ are $\kappa_0$-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1$-adjacent.

**Definition 2.3:** ([1]) Let $a, b \in \mathbb{Z}$, $a < b$. A digital interval is a set of the form

$$[a, b]_Z = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

in which 2-adjacency is assumed.

For example, if $\kappa$ is an adjacency relation on a digital image $Y$, then $f : [a, b]_Z \rightarrow Y$ is $(2, \kappa)$-continuous if and only if for every $\{c, c+1\} \subset [a, b]_Z$, either $f(c) = f(c+1)$ or $f(c)$ and $f(c+1)$ are $\kappa$-adjacent.

### 2.1. Digital Homotopy

In general sense, a homotopy which is defined between continuous maps is a continuous deformation of one into another over a time period.

**Definition 2.4:** ([2]; see also [5]) Suppose that $X \subset \mathbb{Z}^n$ and $Y \subset \mathbb{Z}^n$ be digital images. Let $f, g : X \rightarrow Y$ be $(\kappa, \lambda)$-continuous functions. Assume there is a positive integer $m$ and a function $F : X \times [0, m]_Z \rightarrow Y$ such that

i) for all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x);

ii) for all $x \in X$, the induced function $F_x : [0, m]_Z \rightarrow Y$ defined by

$$F_x(t) = F(x,t) \quad \text{for all } t \in [0, m]_Z$$

is $(2, \lambda)$-continuous.

iii) for all $t \in [0, m]_Z$, the induced function $F_t : [0, m]_Z \rightarrow Y$ defined by

$$F_t(x) = F(x,t) \quad \text{for all } t \in [0, m]_Z$$

is $(\kappa, \lambda)$-continuous.
Then $F$ is called to be a digital $(\kappa, \lambda)$ -- homotopy between $f$ and $g$, and $f$ and $g$ are said to be digitally $(\kappa, \lambda)$ -- homotopic in $Y$.

We use the notation

$$f \approx_{\kappa, \lambda} g$$

to denote $f$ and $g$ are digitally $(\kappa, \lambda)$ -- homotopic in $Y$.

Digital homotopy is an equivalence relation among digitally continuous functions [2, 5]. Furthermore, composition preserves homotopy:

**Proposition 2.5:** ([2]) If $f_0, f_1 : X \rightarrow Y$ are $(\kappa, \lambda)$ -- continuous with $f_0 \approx_{\kappa, \lambda} f_1$ and $g_0, g_1 : Y \rightarrow Z$ are $(\lambda, \mu)$ -- continuous with $g_0 \approx_{\lambda, \mu} g_1$, then $g_0 \circ f_0 \approx_{\lambda, \mu} g_1 \circ f_1$.

**Definition 2.6:** ([3]) Let $f : X \rightarrow Y$ be a $(\kappa, \lambda)$ -- continuous function and $g : Y \rightarrow X$ be a $(\lambda, \kappa)$ -- continuous function such that $f \circ g \approx_{\lambda, \kappa} 1_X$ and $g \circ f \approx_{\kappa, \lambda} 1_Y$.

Then we say $X$ and $Y$ have the same $(\kappa, \lambda)$ -- homotopy type and that $X$ and $Y$ are $(\kappa, \lambda)$ -- homotopy equivalent.

**2.2. Digital Loops**

**Definition 2.8:** ([3]) A digital $\kappa$ -- path in a digital image is a $(2, \kappa)$ -- continuous $\kappa$ -- function $f : [0, m]_Z \rightarrow X$. Also if $f(0) = f(m)$, we say that $f$ is a digital $\kappa$ -- loop, and the point $p = f(0)$ is the base point of the loop $f$. If $f$ is a constant function then it is called a trivial loop.

If $f$ and $g$ are digital $\kappa$ -- paths in $X$ such that $g$ starts where $f$ ends, the product of $f$ and $g$, written $f \cdot g$, is intuitively, the $\kappa$ -- path obtained by following $f$ by $g$. Formally $f : [0, m]_Z \rightarrow X$, $g : [0, m]_Z \rightarrow X$, $f(0) = g(0)$, then $(f \cdot g) : [0, m_1 + m_2]_Z \rightarrow X$ is defined by

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_Z \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_Z \end{cases}.$$ 

**Definition 2.9:** ([3]) Let $f, g : [0, m]_Z \rightarrow (X, x_0)$ be digital loops such that $f(0) = f(m)$, we say that $f$ is a digital $\kappa$ -- loop, and the point $p = f(0)$ is the base point of the loop $f$. If $f$ is a constant function then it is called a trivial loop.

If $H : [0, m]_Z \times [0, m]_Z \rightarrow X$ is a digital homotopy such that for all $t \in [0, m]_Z$ we have

$$H(0, t) = H(m, t) = x_0$$

we say $H$ holds the end points fixed.

Assume that the number $m$ can be consider as the sum of two number formed $m_1 + m_2$. 

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3. DIGITAL HOMOTOPY OF DIGITAL PATHS

Although definitions, propositions and theorems are presented in this section are similar to Boxer [3], here all of them is given on digital homotopy of digital paths.

Before defining the digital fundamental group of a space $X$, we shall consider digital paths on $X$ and an equivalence relation called digital path homotopy between them.

**Definition 3.1:** Let $f$ and $f'$ be $(\kappa, \lambda)$-continuous functions. Two paths $f$ and $f'$, mapping the digital interval $[0, m]_\mathbb{Z}$, are said to be digital path homotopic if they have the same initial point $x_0$ and the same final point $x_1$, and there is a $(\kappa, \lambda)$-continuous map $F : [0, m]_\mathbb{Z} \times [0, m]_\mathbb{Z} \to X$ such that

$$F(s, 0) = f(s) \text{ and } F(s, m) = f'(s).$$

$$F(0, t) = x_0 \text{ and } F(m, t) = x_1$$

for each $s \in [0, m]_\mathbb{Z}$ and each $t \in [0, m]_\mathbb{Z}$. We call $F$ to be a digital path homotopy between $f$ and $f'$, and we write $f \simeq_{p(\kappa, \lambda)} f'$. (See Definition 5.2 of 2)

**Proposition 3.2:** The relation of digital homotopy of digital paths ($\simeq_{p(\kappa, \lambda)}$) is an equivalence relation. (see Proposition 5.2 of 2)

**Proof:**

**Reflexive:** Let the function $f : [0, m]_\mathbb{Z} \to X$, such that

$$0 \mapsto f(0) = x_0$$

$$m \mapsto f(m) = x_1$$

be a digital path. We want to show that $f \simeq_{p(\kappa, \lambda)} f$.

The map defined by $F : [0, m]_\mathbb{Z} \times [0, m]_\mathbb{Z} \to X$ such that

$$(s, t) \mapsto F(s, t) = f(s)$$

is $(\kappa, \lambda)$-continuous since $f$ is $(\kappa, \lambda)$-continuous. Further, since

$$F(s, 0) = f(s), \quad F(s, m) = f(s),$$

$$F(0, t) = f(0) = x_0,$$

$$F(m, t) = f(m) = x_1$$

conditions are provided, hence we get $f \simeq_{p(\kappa, \lambda)} f$.

**Symmetric:** If $f \simeq_{p(\kappa, \lambda)} g$, we want to show that $g \simeq_{p(\kappa, \lambda)} f$ with

$g : [0, m]_\mathbb{Z} \to X$

$$0 \mapsto g(0) = x_0$$

$$m \mapsto g(m) = x_1$$

is a $(\kappa, \lambda)$-continuous function.

$$f \simeq_{p(\kappa, \lambda)} g \iff F : [0, m]_\mathbb{Z} \times [0, m]_\mathbb{Z} \to X$$

$$F(s, 0) = f(s), \quad F(s, m) = g(s)$$

$$F(0, t) = x_0, \quad F(m, t) = x_1$$

where $s, t \in [0, m]_\mathbb{Z}$. 
Define a $G$ map with

$$G : [0,m]_\mathbb{Z} \times [0,m]_\mathbb{Z} \to X$$

$$(s,t) \mapsto G(s,t) = F(s,m-t).$$

We give following conditions:

- $G(s,0) \neq F(s,m) = g(s)$
- $G(s,m) = F(s,0) \neq f(s)$
- $G(0,t) = F(0,m-t) = x_0$
- $G(m,t) = F(m,m-t) = x_1$

$G$ is $(\kappa, \lambda)$-continuous since $F$ is $(\kappa, \lambda)$-continuous. Thus we get $g \equiv_p (\kappa, \lambda) f$.

**Transitivity:** We claim that if $f \equiv_p (\kappa, \lambda) g$ and $g \equiv_p (\kappa, \lambda) h$, then $f \equiv_p (\kappa, \lambda) h$ with

$$h : [0,m]_\mathbb{Z} \to X$$

$$0 \mapsto h(0) = x_0$$

$$m \mapsto h(m) = x_1.$$

where $m = m_1 + m_2$

$$f \equiv_p (\kappa, \lambda) g \iff \exists F : [0,m_1]_\mathbb{Z} \times [0,m_1]_\mathbb{Z} \to X$$

$$F(s,0) \neq f(s), F(s,m_1) = g(s)$$

$$F(0,t) = x_0, \quad F(m_1,t) = x_1,$$

$$g \equiv_p (\kappa, \lambda) h \iff \exists G : [0,m_2]_\mathbb{Z} \times [0,m_2]_\mathbb{Z} \to X$$

$$G(s,0) \neq g(s), G(s,m_2) = h(s)$$

$$G(0,t) = x_0, \quad G(m_2,t) = x_1.$$

The map satisfying above conditions are $(\kappa, \lambda)$-continuous.

$$H : [0,m]_\mathbb{Z} \times [0,m]_\mathbb{Z} \to X$$

$$(s,t) \mapsto H(s,t) = \begin{cases} F(s,t), & \text{if } t \in [0,m_1]_\mathbb{Z} \\ G(s,t-m_1), & \text{if } t \in [m_1,m_1+m_2]_\mathbb{Z} \end{cases}$$

$$H(s,0) = F(s,0) = f(s), \quad H(0,t) = F(0,t) = x_0$$

$$H(0,t) = \begin{cases} F(0,t), & \text{if } t \in [0,m_1]_\mathbb{Z} \\ G(0,t-m_1), & \text{if } t \in [m_1,m_1+m_2]_\mathbb{Z} \end{cases}$$

Hence

$$H(0,t) = x_0 \quad H(s,m) = G(s,m) = h(s), \quad H(m,t) = G(m,t) = x_1$$

$$H(m,t) = \begin{cases} F(m_1,t), & \text{if } t \in [0,m_1]_\mathbb{Z} \\ G(m_2,t-m_1), & \text{if } t \in [m_1,m_1+m_2]_\mathbb{Z} \end{cases}$$

Hence

$$H(m,t) = x_1 \quad H(m,t) = x_1$$

$H$ is $(\kappa, \lambda)$-continuous since $F$ and $G$ are $(\kappa, \lambda)$-continuous. Thus we get $f \equiv_p (\kappa, \lambda) h$.

Finally $\equiv_p (\kappa, \lambda)$ is an equivalence relation.
Proposition 3.3: Suppose that $\kappa$ is an adjacency relation in $X$ and $\lambda$ is an adjacency relation in $Y$. Let $F : X \to Y$ be a digital continuous function. If the digital paths $f, g : [0, m] \to X$ are digital path homotopic in the domain spaces $X$, then the paths $F \circ f, F \circ g : [0, m] \to Y$ are $(\kappa, \lambda)$-homotopic in the codomain space $Y$.

Proof: If $f = p_{(\kappa, \lambda)} g$ in $X$ via the path homotopy $H : [0, m] \times [0, m] \to X$, then $F \circ f = p_{(\kappa, \lambda)} F \circ g$ in $Y$ via the path homotopy $F \circ H : [0, m] \times [0, m] \to X \to Y$.

Definition 3.4: If $f$ is a digital path from $x_0$ to $x_1$ in a digital image $X$ and $g$ is a digital path from $x_1$ to $x_2$ in a digital image $X$, we define the digital product $f \star_{(\kappa, \lambda)} g$ of $f$ and $g$ to be the digital path $h$ by the following form

$$h(s) = \begin{cases} f(s), & \text{for } s \in [0, m_1] \\ g(s - m_1), & \text{for } s \in [m_1, m_1 + m_2]. \end{cases}$$

The function $h$ is well-defined and $(\kappa, \lambda)$-continuous and it is a digital path $x_0$ to $x_2$ in a digital image.

The digital product operation on digital paths induces well defined operation on digital path homotopy classes, defined as below

$$[f] \star_{(\kappa, \lambda)} [g] = [f \star_{(\kappa, \lambda)} g]. \quad [6, \text{Prop 4.8}]$$

Definition 3.5: A digital line is the set of all lattis points in a line in the Euclidean spaces such that $x_1$ and $x_{1+1}$ are adjacent via the adjacency relations defined in the digital image.

Definition 3.6: Let $X$ be a digital image with $X \subset \mathbb{Z}^n$. If for every points $x, y$ of $X$, the digital line segment joining them lies in $X$, that is $(m\frac{t}{m})x + \frac{t}{m} y \in X$ with $m$ is a positive integer, then the digital image $X$ is said to be convex.

Proposition 3.7: Let $X$ be a digital image with $X \subset \mathbb{Z}^n$. If $X$ is a convex set; that is, for every $x, y$ of points of $X$, the digital line segment joining them lies in $X$, then any two paths in $X$ having the same end points are digital path homotopic.

Proof: We want to show that $f = p_{(\kappa, \lambda)} g$ and $f = p_{(\kappa, \lambda)} \alpha \star_{(\kappa, \lambda)} g$ with $f$, $g$ and $\alpha$ indicated as figure.

$$\exists F : [0, m] \times [0, m] \to X, \quad F(s, 0) = f(s), \quad F(s, 1) = \alpha \star_{(\kappa, \lambda)} g(s). \quad \text{We want to check whether following identities are provided:}$$

$$F(0, t) = x_0, \quad F(m, t) = x_1.$$

Here $\alpha \star_{(\kappa, \lambda)} g(s) = \begin{cases} \alpha(s), & \text{for } s \in [0, m_1] \\ g(s - m_1), & \text{for } s \in [m_1, m_1 + m_2]. \end{cases}$
Finally we find $f \simeq p(\kappa, \lambda) g$.

**Proposition 3.8:** Let $X \subset \mathbb{Z}^n$ be a digital image, if $X$ is a convex set, then $X$ is digital path connected.

**Proof:** If $X$ is convex, then for each $t \in [0, m]$, and for every $x, y$ in $X$, we have $(m-t)x + \frac{t}{m} y \in X$. Define a function by $f : [0, m] \rightarrow X$, $f(t) = (m-t)x + \frac{t}{m} y$.

Because of the convexity of digital image $X$, $f$ is a $(\kappa, \lambda)$-continuous function and $f(0) = x, f(m) = y$. Consequently we get that $X$ is digital path connected.

**Definition 3.9:** A space $X \subset \mathbb{Z}^n$ is said to be digitally contractible if the identity map $i_X : X \rightarrow X$ is digitally null homotopic. (see Definition 2.10 of 3)

**Example 3.10:** $I_\mathbb{Z} = [0, m] \subset \mathbb{Z}$ is digitally contractible.

Let $c$ be a constant map from $I_\mathbb{Z}$ to $I_\mathbb{Z}$. We want to show that the identity map $i_i : I_\mathbb{Z} \rightarrow I_\mathbb{Z}$ is digitally homotopic to the constant map $c$ of $I_\mathbb{Z}$. Define

$$H : I_\mathbb{Z} \times I_\mathbb{Z} \rightarrow I_\mathbb{Z}$$

$$(s, t) \mapsto H(s, t) = (\frac{m-t}{m})i_x + \frac{t}{m} c.$$

$$H(s, 0) \simeq i_i, \quad H(s, m) = c.$$

Therefore $I_\mathbb{Z}$ is digitally contractible.

**Theorem 3.11:** A digitally contractible space is digitally path connected. (see Proposition 5.5 of 2)

**Proof:** Let $I_\mathbb{Z} = [0, m] \subset \mathbb{Z}$ and $X \subset \mathbb{Z}^n$ be a digitally contractible space. If $X$ is digitally contractible, then $i_X \simeq p(\kappa, \lambda) c$ where $i_X$ identity map of $X$ and $c$ is constant map of $X$. Thus

$$\exists G : X \times [0, m] \rightarrow X$$

$$(x, t) \mapsto G(x, t) = (\frac{m-t}{m})i_x + \frac{t}{m} c.$$

$$G(x, 0) \simeq i_x, \quad G(x, 1) = c.$$

Where $t \in [0, m]$.

For every elements $x$ and $y$ in $X$, we get

$$\exists f : [0, m] \rightarrow X$$

$$t \mapsto f(t) = (\frac{m-t}{m})x + \frac{t}{m} y \in X.$$

$$G(x, 0) \simeq i_x, \quad G(x, m) = c.$$

$f$ is a digital continuous map. Consequently we see that $f(0) = x$ and $f(1) = y$. Therefore $X$ is digitally path connected.
Theorem 3.12: Let $X$ be a digital image with $X \subset \mathbb{Z}^n$, $I_{\mathbb{Z}} = [0,m]_{\mathbb{Z}}$ a digital interval and $[I_{\mathbb{Z}},X]$ denote the set of digital homotopy classes of maps of $I_{\mathbb{Z}}$ to $X$. If $X$ is digitally path connected, then the set $[I_{\mathbb{Z}},X]$ has a single element.

Proof: Let $X$ be a digital image and $f:I_{\mathbb{Z}} \to X$ and $g:I_{\mathbb{Z}} \to X$ be digitally continuous maps. Since $X$ is digitally path connected, for each $f(s), g(s) \in Y$ and for all $s \in I_{\mathbb{Z}}$, there is a digital continuous map $H:I_{\mathbb{Z}} \to X$ such that $f(s) = H(0)$ and $g(s) = H(1)$. Define a $G$ map such that

$$G : I_{\mathbb{Z}} \times I_{\mathbb{Z}} \to X$$

$$(s,t) \mapsto G(s,t) = H(t).$$

Then we see that $G(S,0) = H(0) = f(s)$ and $G(S,1) = H(1) = g(s)$. $G$ is digitally continuous since $H$ is digitally continuous. Then we have $f \simeq_{(x,y)} g$. Therefore $[I_{\mathbb{Z}},X] = \{[f]\}$.

Corollary 3.13: Let $X$ and $Y$ be digital images. If $X$ is digitally contractible and $Y$ is digitally path connected, then the set $[X,Y]$ has a single element.

Proof: Suppose that $[f],[g] \in [X,Y]$. We want to show that $[f] = [g]$. Since $f, g : X \to Y$ and $Y$ is digitally path connected, there exists a map $\alpha : I_{\mathbb{Z}} \to Y$, $\alpha(0) = f(x)$, $\alpha(m) = g(x)$

$$F : X \times I_{\mathbb{Z}} \to Y$$

$$F(x,s) = \alpha(s)$$

$$F(x,0) = \alpha(0) = f(x)$$

$$F(x,m) = \alpha(m) = g(x)$$

Thus $f = g \Rightarrow [f] = [g]$. Therefore $[X,Y] = \{[f]\}$.

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